VI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

Below is the list of problems for the first (correspondence) round of the VI Sharygin Geometrical Olympiad.

The olympiad is for high-school students of 8–11 grades (these are four elder grades in Russian school). In the list below each problem is indicated by the numbers of school grades, for which it is intended. However, the participants may solve problems for elder grades as well (solutions for younger forms will not be considered).

Your work containing the solutions for the problems (in Russian or in English) should be sent not later than April 1, 2010, by e-mail to geomolymp@mccme.ru in pdf, doc or jpg files. Please, follow several simple rules:

1. Each student sends his work in a separate message.

2. If your work consists of several files, send it as an archive.

3. In the subject of the message write "The work for Sharygin olympiad", and present the following personal information in the letter:

- last name, first name;

- post address, phone number, E-mail;

- the current number of your grade at school;

- the number and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you do not have an e-mail access, please, send your work by regular mail to the following address: *Russia, 119002, Moscow, Bolshoy Vlasyevsky per., 11. Olympiad in honour of Sharygin.* In the title page write your personal information indicated in the item 3 above.

In your work you should start writing the solution to each problem in a new page: first write down the statement of the problem, and then the solution. Present your solutions in detail, including all significant arguments and calculations. Provide all necessary figures. Solutions of computational problems have to be completed with a sharp answer. Please, be accurate to get a good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

Your work will be examined thoroughly, and your marks will be sent to you by the middle of May 2010. Winners of the correspondence round will be invited to take part in the final round in Summer 2010 in Dubna town (near Moscow).

1. (8) Does there exist a triangle, whose side is equal to some of its altitudes, another side is equal to some of its bisectrices, and the third side is equal to some of its medians ?

2. (8) Bisectors AA_1 and BB_1 of a right triangle ABC ($\angle C = 90^\circ$) meet at a point I. Let O be the circumcenter of the triangle CA_1B_1. Prove that $OI \perp AB$.

3. (8) Points A', B', C' lie on sides BC, CA, AB of a triangle ABC. For a point X one has $\angle AXB = \angle A'C'B' + \angle ACB$ and $\angle BXC = \angle B'A'C' + \angle BAC$. Prove that the quadrilateral $XA'BC'$ is cyclic.

4. (8) The diagonals of a cyclic quadrilateral $ABCD$ meet in a point N. The circumcircles of the triangles ANB and CND intersect the sidelines BC and AD for the second time in points A_1, B_1, C_1, D_1. Prove that the quadrilateral $A_1B_1C_1D_1$ is inscribed into a circle centered at N.

5. (8–9) A point E lies on the altitude BD of a triangle ABC, and $\angle AEC = 90°$. Points O_1 and O_2 are the circumcenters of the triangles AEB and CEB; points F, L are the midpoints of the segments AC and O_1O_2. Prove that the points L, E, F are collinear.

6. (8–9) Points M and N lie on the side BC of a regular triangle ABC (M is between B and N), and $\angle MAN = 30°$. The circumcircles of the triangles AMC and ANB meet at a point K. Prove that the line AK passes through the circumcenter of the triangle AMN.

7. (8–9) The line passing through the vertex B of a triangle ABC and perpendicular to its median BM intersects the altitudes dropped from A and C (or their extensions) in points K and N. Points O_1 and O_2 are the circumcenters of the triangles ABK and CBN respectively. Prove that $O_1M = O_2M$.

8. (8–10) Let AH be the altitude of a given triangle ABC. Points I_b and I_c are the incenters of the triangles ABH and ACH respectively; BC touches the incircle of ABC at a point L. Find $\angle LI_bI_c$.

9. (8–10) A point inside a triangle is called "good" if three cevians passing through it are equal. Assume for a isosceles triangle ABC ($AB = BC$) the total number of good points is odd. Find all possible values of this number.

10. (8–11) Let three lines forming a triangle ABC be given. Using a two-sided ruler and drawing at most eight lines construct a point D on the side AB such that $AD/BD = BC/AC$.

11. (8–11) A convex n-gon is split into three convex polygons. One of them has n sides, the second one has more than n sides, the third one has less than n sides. Find all possible values of n.

12. (9) Let AC be the greatest leg of a right triangle ABC, and CH be the altitude to its hypothenuse. The circle of radius CH centered at H intersects AC in point M. Let a point B' be the reflection of B with respect to the point H. The perpendicular to AB erected at B' meets the circle in a point K. Prove that:

    a) $B'M \parallel BC$;

    b) AK is tangent to the circle.

13. (9) Let us have a convex quadrilateral $ABCD$ such that $AB = BC$. A point K lies on the diagonal BD, and $\angle AKB + \angle BKC = \angle A + \angle C$. Prove that $AK \cdot CD = KC \cdot AD$.

14. (9–10) We have a convex quadrilateral $ABCD$ and a point M on its side AD such that CM and BM are parallel to AB and CD respectively. Prove that $S_{ABCD} \geq 3S_{BCM}$.

15. (9–11) Let AA_1, BB_1 and CC_1 be the altitudes of an acute-angled triangle ABC, AA_1 meets B_1C_1 in a point K. The circumcircles of the triangles A_1KC_1 and A_1KB_1 intersect the lines AB and AC for the second time at points N and L respectively. Prove that

a) the sum of diameters of these two circles is equal to BC;

b) $A_1N/BB_1 + A_1L/CC_1 = 1$.

16. (9–11) A circle touches the sides of an angle with vertex A at points B and C. A line passing through A intersects this circle in points D and E. A chord BX is parallel to DE. Prove that XC passes through the midpoint of the segment DE.

17. (9–11) Construct a triangle, if the lengths of the bisectrix and of the altitude from one vertex, and of the median from another vertex are given.

18. (9–11) A point B lies on a chord AC of a circle ω. Segments AB and BC are diameters of circles ω_1 and ω_2 centered at O_1 and O_2 respectively. These circles intersect ω for the second time in points D and E respectively. The rays O_1D and O_2E meet in a point F, and the rays AD and CE do in a point G. Prove that the line FG passes through the midpoint of the segment AC.

19. (9–11) A quadrilateral $ABCD$ is inscribed into a circle with center O. Points P and Q are opposite to C and D respectively. Two tangents drown to that circle at these points meet the line AB in points E and F (A is between E and B, B is between A and F). The line EO meets AC and BC in points X and Y respectively, and the line FO meets AD and BD in points U and V. Prove that $XV = YU$.

20. (10) The incircle of an acute-angled triangle ABC touches AB, BC, CA at points C_1, A_1, B_1 respectively. Points A_2, B_2 are the midpoints of the segments B_1C_1, A_1C_1 respectively. Let P be a common point of the incircle and the line CO, where O is the circumcenter of ABC. Let also A' and B' be the second common points of PA_2 and PB_2 with the incircle. Prove that a common point of AA' and BB' lies on the altitude of the triangle dropped from the vertex C.

21. (10–11) A given convex quadrilateral $ABCD$ is such that $\angle ABD + \angle ACD > \angle BAC + \angle BDC$. Prove that $S_{ABD} + S_{ACD} > S_{BAC} + S_{BDC}$.

22. (10–11) A circle centered at a point F and a parabola with focus F have two common points. Prove that there exist four points A, B, C, D on the circle such that the lines AB, BC, CD and DA touch the parabola.

23. (10–11) A cyclic hexagon $ABCDEF$ is such that $AB \cdot CF = 2BC \cdot FA$, $CD \cdot EB = 2DE \cdot BC$, and $EF \cdot AD = 2FA \cdot DE$. Prove that the lines AD, BE and CF concur.

24. (10–11) Let us have a line l in the space and a point A not lying on l. For an arbitrary line l' passing through A, XY (Y is on l') is a common perpendicular to the lines l and l'. Find the locus of points Y.

25. (11) For two different regular icosahedrons it is known that some six of their vertices are vertices of a regular octahedron. Find the ratio of the edges of these icosahedrons.