XI Geometrical Olympiad in honour of I.F.Sharygin
Final round. Grade 8. First day. Solutions

Ratmino, 2015, July 30.

1. (V. Yasinsky) In trapezoid $ABCD$ angles A and B are right, $AB = AD$, $CD = BC + AD$, $BC < AD$. Prove that $\angle ADC = 2\angle ABE$, where E is the midpoint of segment D.

   First solution. Choose a point K on the side CD so that $CK = CB$; let M be the common point of AB and the perpendicular from K to CD. The right triangles BCM and KCM are congruent by hypothenuse and leg, so $BM = MK$, and CM bisects the angle C. By the problem condition we get $KD = AD$, and in a similar way we obtain that $AM = MK$ and that DM bisects the angle D. Thus $AM = AB/2 = AE$, i.e., the triangles ABE and ADM are congruent. Therefore, $\angle ADC = 2\angle ADM = 2\angle ABE$ (Fig. 8.1).

   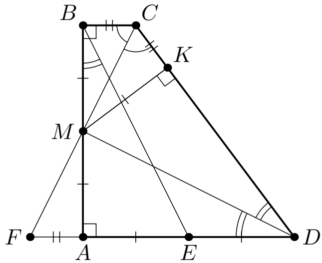

   Fig. 8.1

   Second solution. Choose a point F on the extension of DA beyond point A so that $AF = BC$. Then we have $DF = DC$. Let M be the common point of AB and CF, i.e., M is the midpoint of AB. The right triangles ABE and ADM are congruent, and DM is a median in the triangle CDF, so it bisects the angle CDA; therefore, $\angle CDA/2 = \angle ABE$.

2. (A. Blinkov) A circle passing through A, B and the orthocenter of triangle ABC meets sides AC, BC at their inner points. Prove that $60^\circ < \angle C < 90^\circ$.

   First solution. Let A' and B' be the second meeting points of the circle with BC and AC, respectively. Then $\angle C = (\smile AB - \smile A'B')/2$. Since the angle between the altitudes is equal to $180^\circ - \angle C = \smile AB/2$, we obtain that $180^\circ - \angle C > \angle C$ and thus $\angle C < 90^\circ$.

   On the other hand, the angle C is greater than the angle between the tangents to the circle at A and B; the latter angle is equal to $180^\circ - 2\angle C$. Therefore, $\angle C > 60^\circ$.

   Second solution. If the angle C is not acute, then H either lies outside the triangle or coincides with C. In both cases the intersection points do not belong to the interiors of the sides.

Since $\angle AA'B = \angle BB'A = \angle AHB = 180° - \angle C$, we have $\angle AA'C = \angle BB'C = \angle C$; but these angles are greater than the angles A and B as they are external angles of triangles $AA'B$ and $BB'A$. Therefore, C is the largest angle of the triangle ABC, i.e. $\angle C > 60°$.

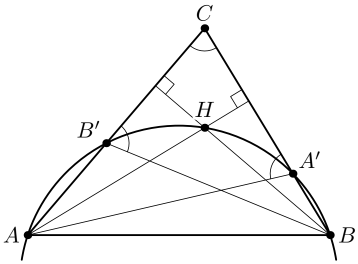

Fig 8.2

3. (M. Yevdokimov) In triangle ABC we have $AB = BC$, $\angle B = 20°$. Point M on AC is such that $AM : MC = 1 : 2$, point H is the projection of C to BM. Find angle AHB.

   Answer. $100°$.

   First solution. Construct a point D such that the quadrilateral $ABCD$ is a rhombus. Let O be the center of the rhombus. Then the line BM divides the median AO of the triangle ABD in ratio $2 : 1$. Thus this line is also a median, i.e., it passes through the midpoint K of the segment AD. Since the points O and H lie on the circle with diameter BC, we have $\angle KHO = \angle BCO = \angle KAO$. Therefore, the quadrilateral $AHOK$ is cyclic, so $\angle AHK = \angle AOK = 80°$, and hence $\angle AHB = 100°$ (Fig. 8.3).

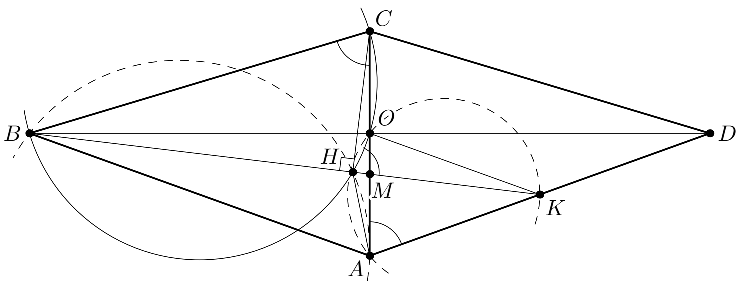

Fig. 8.3

   Second solution. As in the previous solution, we notice that the quadrilateral $BCOH$ is cyclic. So, $MH \cdot MB = MO \cdot MC = MA^2$. Thus the circle AHB is tangent to the line AC, which yields $\angle AHB = 180° - \angle BAC = 100°$.

4. (N. Belukhov) Prove that an arbitrary convex quadrilateral can be divided into five polygons having symmetry axes.

   Solution. Let $ABCD$ be a given quadrilateral. If $AB + CD = AD + BC$, then the quadrilateral is circumscribed. The radii of its incircle joining the center with the points of tangency with the sides divide the quadrilateral into four symmetric quadrilaterals. It remains to divide one of them into two isosceles triangles.

Now assume that $AB + CD > AD + BC$. Construct a circle with center O_1 tangent to sides AB, AD, and CD at points P_1, Q_1, and R_1 respectively, and also construct a circle with center O_2 tangent to the sides AB, BC, and CD at points P_2, Q_2, R_2 respectively (Fig. 8.4). The radii passing to these six points divide $ABCD$ into the quadrilaterals $AP_1O_1Q_1$, $BP_2O_2Q_2$, $CQ_2O_2R_2$, $DQ_1O_1R_1$ and the hexagon $P_1P_2O_2R_2R_1O_1$. Their symmetry axes are respectively the bisectors of the angles A, B, C, D, and the bisector of the angle between the lines AB and CD.

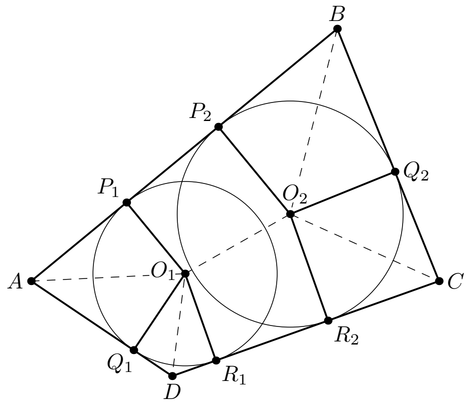

Fig. 8.4

XI Geometrical Olympiad in honour of I.F.Sharygin
Final round. Grade 8. Second day

Ratmino, 2015, July 31.

5. (E. Bakayev, A. Zaslavsky) Two equal hard triangles are given. One of their angles is equal to α (these angles are marked). Dispose these triangles on the plane in such a way that the angle formed by some three vertices would be equal to $\alpha/2$. (*No instruments are allowed, even a pencil.*)

   Solution. The required configuration is shown in Fig 8.5. The triangle BCC' is isosceles, with $BC = C'C$ and $\angle C'CB = 180° - \alpha$. Thus $\angle C'BC = \alpha/2$.

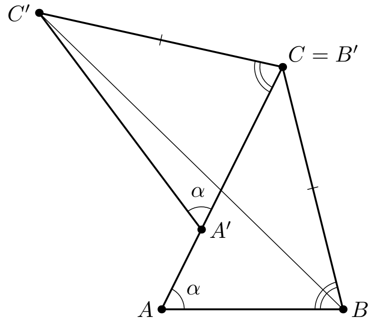

Fig. 8.5

6. (D. Prokopenko) Lines b and c passing through vertices B and C of triangle ABC are perpendicular to sideline BC. The perpendicular bisectors to AC and AB meet b and c at points P and Q respectively. Prove that line PQ is perpendicular to median AM of triangle ABC.

   First solution. Let M be the midpoint of BC. It suffices to prove that $AP^2 - AQ^2 = MP^2 - MQ^2$.

   Since the points P and Q lie on the perpendicular bisectors to AC and BC, respectively, we have $AP^2 - AQ^2 = CP^2 - BQ^2 = (BC^2 + BP^2) - (BC^2 + CQ^2) = (MB^2 + BP^2) - (MC^2 + CQ^2) = MP^2 - MQ^2$.

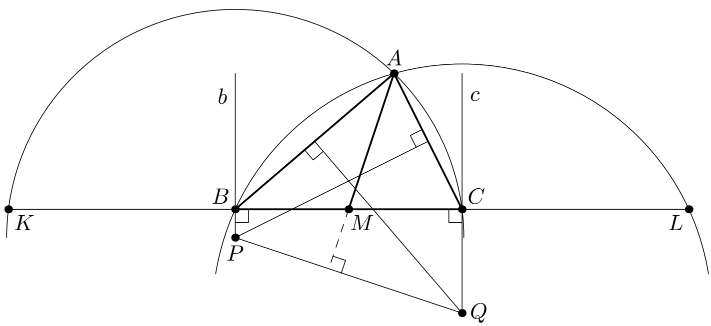

Fig. 8.6

Second solution. Construct a circle centered at P and passing through A. It meets BC at C and also at a point K symmetric to C in B. Similarly, the circle centered at Q and passing through A meets BC at B and at a point L symmetric to B in C. The powers of M with respect to these circles are equal, so the radical axis of these circles is AM, and it is perpendicular to the line of centers PQ (Fig. 8.6).

7. (M. Kungozhin) Point M on side AB of quadrilateral $ABCD$ is such that quadrilaterals $AMCD$ and $BMDC$ are circumscribed around circles centered at O_1 and O_2 respectively. Line O_1O_2 cuts an isosceles triangle with vertex M from angle CMD. Prove that $ABCD$ is a cyclic quadrilateral.

   Solution. If $AB \parallel CD$ then the incircles of $AMCD$ and $BMDC$ have equal radii; now the problem conditions imply that the whole picture is symmetric about the perpendicular from M to O_1O_2, and hence $ABCD$ is an iosceles trapezoid (or a rectangle).

   Now suppose that the lines AB and CD meet at a point K; we may assume that A lies between K and B. The points O_1 and O_2 lie on the bisector of the angle BKC. By the problem condition, this angle bisector forms equal angles with the lines CM and DM; this yields $\angle DMK = \angle KCM$ (Fig. 8.7). Since O_1 and O_2 are the incenter of $\triangle KMC$ and an excenter of $\triangle KDM$, respectively, we have $\angle DO_2K = \angle DMK/2 = \angle KCM/2 = \angle DCO_1$, so the quadrilateral CDO_1O_2 is cyclic. Next, the same points are an excenter of $\triangle AKD$ and the incenter of $\triangle KBC$, respectively, so $\angle KAD = 2\angle KO_1D = 2\angle DCO_2 = \angle KCB$; this implies the desired cyclicity of the quadrilateral $ABCD$.

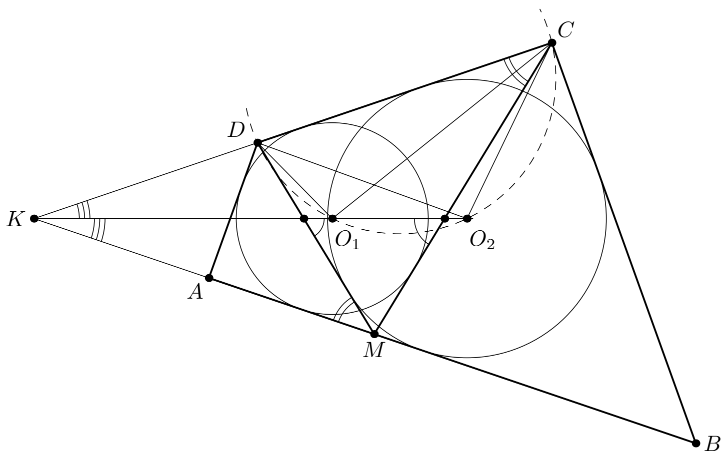

Fig. 8.7

8. (A. Antropov, A. Yakubov) Points C_1, B_1 on sides AB, AC respectively of triangle ABC are such that $BB_1 \perp CC_1$. Point X lying inside the triangle

is such that $\angle XBC = \angle B_1BA$, $\angle XCB = \angle C_1CA$. Prove that $\angle B_1XC_1 = 90^\circ - \angle A$.

First solution. Let X_1 be the projection of X onto BC, and let O be the meeting point of the lines BB_1 and CC_1. Then the triangle C_1BO is similar to the triangle XBX_1 by two angles, thus $\frac{BC_1}{BX} = \frac{BO}{BX_1}$. This implies that the triangles BC_1X and BOX_1 are also similar by two proportional sides and the angle between them. Therefore, $\angle BXC_1 = \angle BX_1O$. Similarly we obtain $\angle B_1XC = \angle OX_1C$ (Fig. 8.8). Hence $\angle BXC_1 + \angle CXB_1 = \angle BX_1O + \angle OX_1C = 180^\circ$. Therefore, $\angle C_1XB_1 = 180^\circ - \angle BXC = \angle XBC + \angle XCB = \angle ABB_1 + \angle ACC_1 = \angle BOC - \angle BAC = 90^\circ - \angle A$, as required.

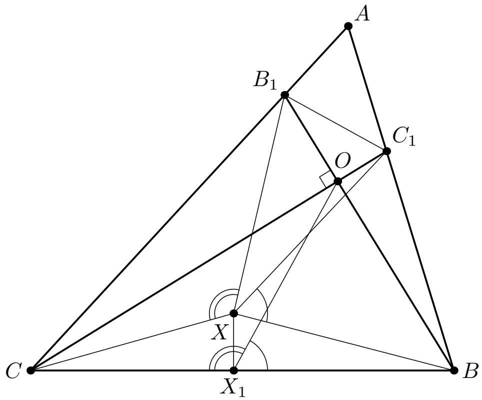

Fig. 8.8

Second solution. We start with two lemmas.

Lemma 1. If the diagonals of a quadrilateral are perpendicular, then the projections of their meeting point to the sidelines are concyclic.

Proof. Let the diagonals of a quadrilateral $ABCD$ meet at O, and let K, L, M, and N be the projections of O onto AB, BC, CD, and DA, respectively. Since the quadrilaterals $OKBL$, $OLCM$, $OMDN$, and $ONAK$ are cyclic, we have $\angle LKN + \angle LMN = \angle OBC + \angle OCB + \angle OAD + \angle ODA = 180^\circ$.

Lemma 2. If the projections of a point P onto the sidelines of $ABCD$ are concyclic, then the reflections of the lines AP, BP, CP, and DP about the bisectors of the corresponding angles are concurrent.

Proof. Since the projections of P to the sidelines are concyclic, the points K, L, M, and N which are symmetric to P about the lines AB, BC, CD, and DA, respectively, are also concyclic. Since $AK = AP = AN$, the perpendicular bisector to the segment KN coincides with the bisectrix of the angle KAN; this bisectrix is symmetric to AP about the bisectrix of the angle BAD. Therefore, all of these four lines pass through the circumcenter of $KLMN$.

To solve the problem, apply these two lemmas to the quadrilateral BCC_1B_1. Since the lines BX and CX are symmetric to BB_1 and CC_1 about the bisectors of the angles B and C respectively, the lines B_1X and C_1X are also symmetric to B_1B and C_1C about the bisectors of the angles CB_1C_1 and BC_1B_1 respectively. This yields that $\angle B_1XC_1 = 180° - \angle XB_1C_1 - \angle XC_1B_1 = 180° - \angle CB_1O - \angle BC_1O = 180° - (90° + \angle A) = 90° - \angle A$.

XI Geometrical Olympiad in honour of I.F.Sharygin
Final round. Grade 9. First day
Ratmino, 2015, July 30.

1. (D. Mukhin) Circles α and β pass through point C. The tangent to α at this point meets β at point B, and the tangent to β at C meets α at point A so that A and B are distinct from C and angle ACB is obtuse. Line AB meets α and β for the second time at points N and M respectively. Prove that $2MN < AB$.

   Solution. Since AC and BC are tangent to β and α, respectively, we have $\angle ACM = \angle CBA$ and $\angle BCN = \angle CAB$. Since the angle ACB is obtuse, the points A, M, N, B are arranged on AB in this order. Using the tangency again, we get $AM = AC^2/AB$ and $BN = BC^2/AB$. Using the AM–QM inequality and the triangle inequality in this order, we obtain $2(AM+BN) \geqslant \frac{(AC+BC)^2}{AB} > AB$; this is equivalent to the desired inequality.

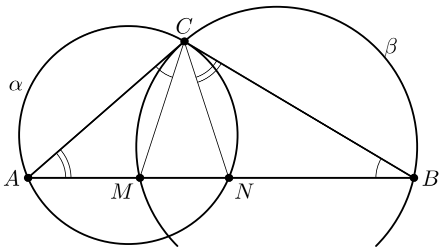

Fig. 9.1

2. (A. Zaslavsky) A convex quadrilateral is given. Using a compass and a ruler construct a point such that its projections to the sidelines of this quadrilateral are the vertices of a parallelogram.

   Solution. All the angles in the solution are directed.

   Let K, L, M, and N be the projections of the point P onto AB, BC, CD, and DA, respectively. The condition $KL \parallel MN$ is equivalent to $\angle BKL + \angle MND = \angle BAD$. Since $PKBL$ and $PMDB$ are cyclic quadrilaterals, we have $\angle BKL = \angle BPL$ and $\angle MND = \angle MPD$. Consequently, the condition $KL \parallel MN$ is equivalent to

$$\angle BPD = (\angle BPL + \angle MPD) + \angle LPM = \angle BAD + (180^\circ - \angle DCB).$$

   Thus, we can construct a circle passing through B and D and containing P (Fig. 9.2).

   Similarly, the condition $KN \parallel LM$ is equivalent to $\angle CPA = 180^\circ + \angle CBA - \angle ADC$, so we can construct a circle through A and C containing P. One of the meeting points of the two constructed circles is the Miquel point of the lines AB, BC, CD, and DA (its projections are collinear by the Simson theorem). The other point is the desired one.

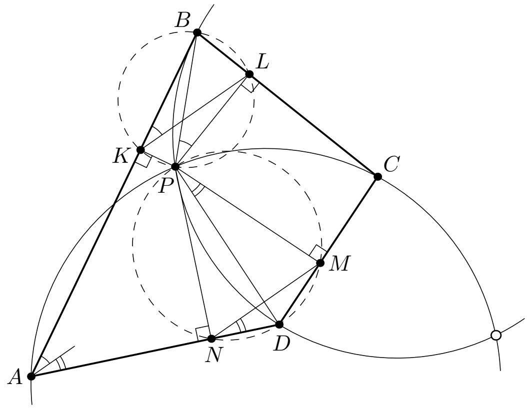

Fig. 9.2

3. (M. Kharitonov, A. Polyansky) Let 100 discs lie on the plane in such a way that each two of them have a common point. Prove that there exists a point lying inside at least 15 of these discs.

   Solution. Let K be the smallest of the given discs; we may suppose that its radius is equal to 1. Let O be the center of K, and let $A_1A_2A_3A_4A_5A_6$ be a regular hexagon with center O and side length $\sqrt{3}$. We will prove that each of given discs contains one of points $O, A_1, \ldots, A_6$; by pigeonhole principle, this implies the problem statement.

   Let O' be the center of some disc K'. If O' lies in K, then K' contains O, because the radius of K' is at least 1. So we assume henceforth that $OO' > 1$.

   The angle between the ray OO' and one of the rays OA_i (say, with OA_1) is at most 30°. Thus we have

$$O'A_1^2 = O'O^2 + OA_1^2 - 2O'O \cdot OA_1 \cos \angle O'OA_1 \leqslant O'O^2 + 3 - 3O'O.$$

   If $1 < O'O \leq 2$, then $O'A_1 \leq 1$, so K' contains A_1. Otherwise, we have $O'O > 2$, and this implies that $O'A_1 < O'O - 1$. On the other hand, the radius of K' is not less than $OO' - 1$ because this disc intersects K, so in this case K' also contains A_1.

4. (R. Krutovsky, A. Yakubov) A fixed triangle ABC is given. Point P moves on its circumcircle so that segments BC and AP intersect. Line AP divides triangle BPC into two triangles with incenters I_1 and I_2. Line I_1I_2 meets BC at point Z. Prove that all lines ZP pass through a fixed point.

   First solution. It is known that the centers of two arbitrary circles, together with their internal and external homothety centers, form a harmonic quadruple. For two given circles centered at I_1 and I_2, the external homothety center is Z, and the internal one lies on the line AP, since BZ and AP are an external and an internal common tangents, respectively. Thus, the projection mapping centered at P acting from the line I_1I_2 to the circumcircle Ω of $\triangle ABC$ maps the internal homothety center to A, and the points I_1 and I_2 to the midpoints of the arcs AB and AC, respectively. These three points are fixed, so the image of Z is also fixed, and all lines PQ pass through this fixed point on Ω.

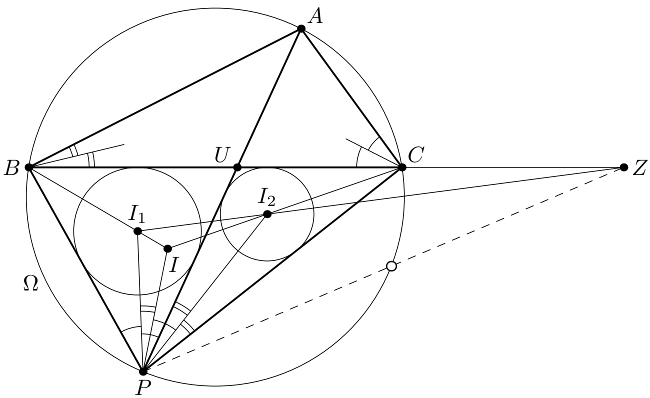

Fig. 9.4

Second solution. Let U be the common point of AP and BC. We prove that the cross-ratio $(BC;ZU)$ is independent of P; again, after projecting the line BC from P to the circumcircle Ω of $\triangle ABC$ this will yield that the line PZ meets Ω at a fixed point.

Let I, I_1, and I_2 be the incenters of the triangles PBC, PBU, and PCU respectively. Applying Menelaus' theorem to the triangle BIC we obtain

$$\frac{BZ}{CZ} = \frac{BI_1}{I_1I} \cdot \frac{II_2}{I_2C}.$$

Since PI, PI_1, and PI_2 are the bisectors of the angles BPC, BPU, and CPU respectively, we have $\angle BPI_1 = \angle IPI_2 = \angle C/2$ and $\angle I_1PI = \angle I_2PC = \angle B/2$. Hence, applying the sine law to the triangles BPI_1, I_1PI, IPI_2, and I_2PC we obtain

$$\frac{BI_1}{I_1I} = \frac{BP}{IP} \cdot \frac{\sin(\angle C/2)}{\sin(\angle B/2)} \quad \text{and} \quad \frac{II_2}{I_2C} = \frac{IP}{CP} \cdot \frac{\sin(\angle C/2)}{\sin(\angle B/2)}.$$

Similarly, applying the sine law to the triangles BPC, ABU, and ACU, we get

$$\frac{BP}{CP} = \frac{\sin\angle BCP}{\sin\angle CBP} = \frac{\sin\angle BAU}{\sin\angle CAU} = \frac{BU}{AB} \cdot \frac{AC}{CU}.$$

Multiplying the four obtained equalities, we conclude that

$$(BC;ZU) = \frac{\sin^2(\angle C/2)}{\sin^2(\angle B/2)} \cdot \frac{AC}{AB} = \frac{\operatorname{tg}(\angle C/2)}{\operatorname{tg}(\angle B/2)}.$$

Remark. The value of $(BC; ZU)$ obtained above may be implemented to show that the line PZ meets the circumcircle at the point collinear with the incenter of $\triangle ABC$ and the midpoint of the arc CAB. One may also see that there exists a circle tangent to the circumcircle at that point and also tangent to the segments AB and AC.

XI Geometrical Olympiad in honour of I.F.Sharygin
Final round. Grade 9. Second day
Ratmino, 2015, July 31.

5. (D. Svhetsov) Let BM be a median of nonisosceles right-angled triangle ABC ($\angle B = 90^\circ$), and H_a, H_c be the orthocenters of triangles ABM, CBM respectively. Prove that lines AH_c and CH_a meet on the medial line of triangle ABC.

   Solution. Let A' and C' be the midpoints of the legs AB and BC, respectively. Since the triangles AMB and BMC are isosceles, their altitudes from M pass through A' and C', respectively. Then $AA' \perp BC \perp H_cC'$, $AH_a \perp BM \perp H_cC$, and $A'H_a \perp AB \perp C'C$. This shows that the corresponding sides of the triangles $AA'H_a$ and $H_cC'C$ are parallel, i.e., these triangles are homothetic. Therefore, the lines AH_c, H_aC, and $A'C'$ concur at the homothety center, and it lies on the midline $A'C'$ (Fig. 9.5).

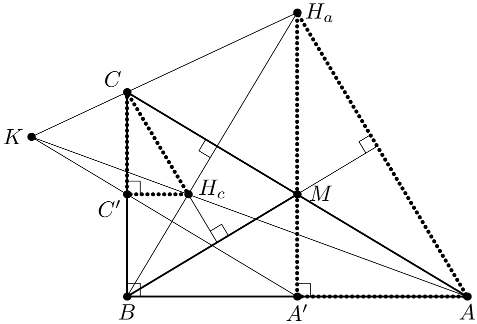

Fig. 9.5

6. (A. Zaslavsky) The diagonals of convex quadrilateral $ABCD$ are perpendicular. Points A', B', C', D' are the circumcenters of triangles ABD, BCA, CDB, DAC respectively. Prove that lines AA', BB', CC', DD' concur.

   Solution. We start with two lemmas.

   Lemma 1. If the diagonals of a quadrilateral are perpendicular, then the projections of their meeting point to the sidelines are concyclic.

   Proof. Let the diagonals of a quadrilateral $ABCD$ meet at O, and let K, L, M, and N be the projections of O onto AB, BC, CD, and DA, respectively. Since the quadrilaterals $OKBL$, $OLCM$, $OMDN$, and $ONAK$ are cyclic, we have $\angle LKN + \angle LMN = \angle OBC + \angle OCB + \angle OAD + \angle ODA = 180^\circ$.

   Lemma 2. If the projections of a point P onto the sidelines of $ABCD$ are concyclic, then the reflections of the lines AP, BP, CP, and DP about the bisectors of the corresponding angles are concurrent.

Proof. Since the projections of P to the sidelines are concyclic, the points K, L, M, and N which are symmetric to P about the lines AB, BC, CD, and DA, respectively, are also concyclic. Since $AK = AP = AN$, the perpendicular bisector to the segment KN coincides with the bisectrix of the angle KAN; this bisectrix is symmetric to AP about the bisectrix of the angle BAD. Therefore, all of these four lines pass through the circumcenter of $KLMN$.

Now we pass to the solution. Note that the line AC contains the altitude of the triangle DAB, so the line AA' is its reflection about the bisector of the angle A; similar arguments work for the other vertices of $ABCD$. Thus the four lines under consideration are concurrent.

Remark. One may show that, if three of the lines AA', BB', CC', and DD' are concurrent, then either the quadrilateral $ABCD$ is cyclic or its diagonals are perpendicular. In both cases the fourth line also passes through the concurrency point.

7. (D. Krekov) Let ABC be an acute-angled, nonisosceles triangle. Altitudes AA' and BB' meet at point H, and the medians of triangle AHB meet at point M. Line CM bisects segment $A'B'$. Find angle C.

Answer. $45°$.

Solution. Let C_0 be the midpoint of AB, and let H' be the point symmetric to H in C_0; it is well known that H' is the point opposite to C on the circumcircle of $\triangle ABC$. The medians CC_0 and CM of similar triangles ABC and $A'B'C$ are symmetric about the bisector of the angle C. An altitude CH and the diameter CH' of the circumcircle are also symmetric about this bisector. Therefore, $\angle H'CC_0 = \angle MCH$, i.e., CM is a symmedian in the triangle CHH' (Fig. 9.7). Thus we have $(CH'/CH)^2 = H'M/MH = 2$; now, from $CH = CH' \cos \angle C$ we obtain $\angle C = 45°$.

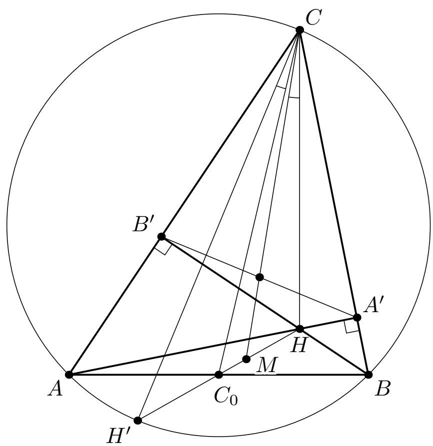

Fig. 9.7

8. (I.Frolov) A perpendicular bisector to side BC of triangle ABC meets lines AB and AC at points A_B and A_C respectively. Let O_a be the circumcenter of triangle AA_BA_C. Points O_b and O_c are defined similarly. Prove that the circumcircle of triangle $O_aO_bO_c$ touches the circumcircle of the original triangle.

   Solution. Let A', B', and C' be the pairwise meeting points of the tangents to the circumcircle of $\triangle ABC$ at its vertices (so, A' is the meeting point of the tangents at B and C, and so on).

   Consider an arbitrary triangle formed by the lines CA, CB, and any line ℓ perpendicular to AB. All such triangles are pairwise homothetic at C. Moreover, if the line ℓ moves with a constant speed, then the circumcenter of the triangle also moves with a constant speed along some line passing through C.

   Consider two specific positions of ℓ, when it passes through B and A. For the first position, the circumcenter of the obtained triangle CC'_AB is A', since $CA' = BA'$ and $\angle CA'B = 180° - 2\angle A = 2\angle CC'_AB$ (Fig. 9.8). Similarly, the circumcenter for the second position in B'. Hence, the circumcenter O_c of the triangle CC_AC_B is the midpoint of $A'B'$. (Different cases of mutual positions of the points can be treated analogously.)

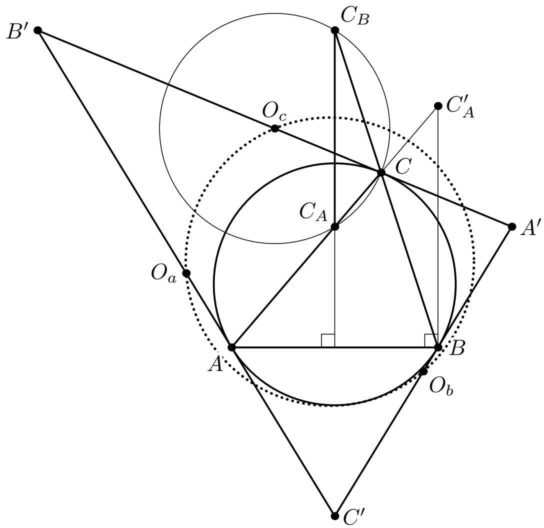

Fig. 9.8

   In a similar way we obtain that O_a and O_b are the midpoints of $B'C'$ and $C'A'$ respectively. Therefore, the circumcircle of $\triangle O_aO_bO_c$ is the Euler circle of the triangle $A'B'C'$, and the circumcircle of $\triangle ABC$ is either its incircle (if $\triangle ABC$ is acute-angled) or its excircle (otherwise). In any case, the Feuerbach theorem shows that these two circles are tangent to each other.

XI Geometrical Olympiad in honour of I.F.Sharygin
Final round. Grade 10. First day

Ratmino, 2015, July 30.

1. (A. Karlyuchenko) Let K be an arbitrary point on side BC of triangle ABC, and KN be a bisector of triangle AKC. Lines BN and AK meet at point F, and lines CF and AB meet at point D. Prove that KD is a bisector of triangle AKB.

   Solution. By the bisector property, we have $\frac{CN}{NA} = \frac{CK}{KA}$. Now by Ceva's theorem we obtain
   $$\frac{BD}{DA} = \frac{BK}{KC} \cdot \frac{CN}{NA} = \frac{BK}{KA},$$
   which means exactly that KD is a bisector of the angle AKB.

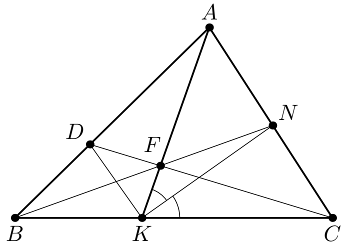

Fig. 10.1

2. (A. Shapovalov) Prove that an arbitrary triangle with area 1 can be covered by an isosceles triangle with area less than $\sqrt{2}$.

   Solution. Let ABC be a given triangle with $AB \geq AC \geq BC$. Let CH be its altitude. Let A' be the reflection of A in H, and let B' be the reflection of B about the bisector of angle A. Then each of the isosceles triangles ACA' and ABB' covers $\triangle ABC$, and we have $S_{ACA'}/S_{ABC} = AA'/AB = 2AH/AB$ and $S_{ABB'}/S_{ABC} = AB'/AC = AB/AC$. The product of these two ratios is $2AH/AC < 2$; therefore, one of them is less than $\sqrt{2}$.

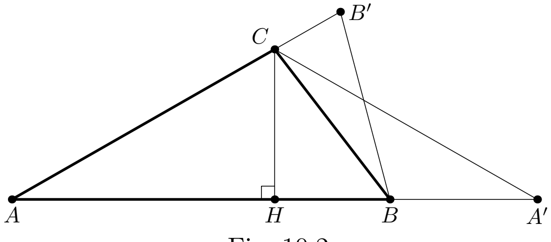

Fig. 10.2

3. (A. Akopyan) Let A_1, B_1 and C_1 be the midpoints of sides BC, CA and AB of triangle ABC. Points B_2 and C_2 are the midpoints of segments BA_1 and CA_1 respectively. Point B_3 is symmetric to C_1 wrt B, and C_3 is symmetric

to B_1 wrt C. Prove that one of common points of circles BB_2B_3 and CC_2C_3 lies on the circumcircle of triangle ABC.

First solution. Choose a point X on the circumcircle Ω of $\triangle ABC$ so that $\angle XA_1C = \angle CA_1A$. Then X is symmetric to the second common point of AA_1 and Ω about the perpendicular bisector of BC; hence, $A_1X \cdot A_1A = A_1B \cdot A_1C = A_1C^2$. This implies that the triangles XA_1C and CA_1A are similar.

Let T be the midpoint of AA_1. Then XC_2 and CT are corresponding medians in similar triangles, thus $\angle CXC_2 = \angle ACT$ (Fig. 10.3.1). On the other hand, the quadrilateral CTC_2C_3 is a parallelogram, i.e., $\angle CC_3C_2 = \angle ACT = \angle CXC_2$. Hence, X lies on the circumcircle of $\triangle CC_2C_3$.

Similarly, X lies on the circumcircle of $\triangle BB_2B_3$. Therefore, X is a desired point.

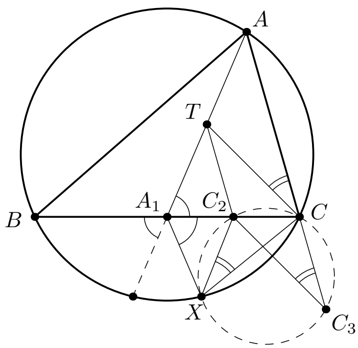

Fig. 10.3.1

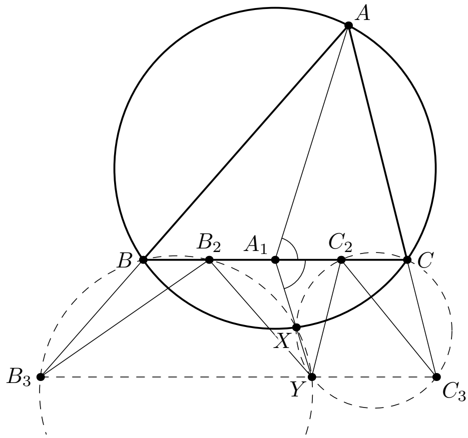

Fig. 10.3.2

Second solution. Let us construct a triangle YB_2C_2 similar to $\triangle ABC$ and separated from that by the line BC (Fig, 10.3.2). Since $B_2C_2 = BC/2$, the points B_3, Y, and C_3 are equidistant from the line BC (this common distance is half the distance of A from BC). Moreover, we have $YB_2 = AB/2 = BB_3$ and $YC_2 = AC/2 = CC_3$. Altogether this means that BB_2YB_3 and CC_2YC_3 are isosceles trapezoids, and hence Y is one of the common points of the circles BB_2B_3 and CC_2C_3.

Since the powers of A_1 with respect to these circles are equal, their second common point lies on A_1Y. Let the ray A_1Y meet the circumcircle of $\triangle ABC$ at X. Since AA_1 and YA_1 are corresponding medians in similar triangles ABC and YB_2C_2, we obtain $\angle XA_1C = \angle CA_1A$; thus, as in the previous solution, we have $A_1X \cdot A_1A = A_1B \cdot A_1C$. Therefore, $A_1X \cdot A_1Y = A_1X \cdot A_1A/2 = A_1B^2/2 = A_1B \cdot A_1B_2$, and so X is the second common point of the two circles under consideration.

4. (I. Yakovlev) Let AA_1, BB_1, CC_1 be the altitudes of an acute-angled, non-isosceles triangle ABC, and A_2, B_2, C_2 be the touching points of sides BC, CA, AB with the correspondent excircles. It is known that line B_1C_1 touches the incircle of ABC. Prove that A_1 lies on the circumcircle of $A_2B_2C_2$.

First solution. Let H, I, and O be the orthocenter, the incenter, and the circumcenter of $\triangle ABC$, respectively. Let Ω, ω, and r denote its circumcircle, incircle, and inradius, respectively. Let A', B', and C' be the tangency points of ω with the sides BC, AC, and AB, and let I_A, I_B, and I_C be the excenters corresponding to these sides, respectively. Finally, let M_A be the midpoint of BC.

The problem condition says that the quadrilateral BC_1B_1C is circumscribed around ω. This quadrilateral is also inscribed into a circle with diameter BC. It is known that in such a quadrilateral, the circumcenter, the incenter, and the meeting point of the diagonals are collinear (e.g., this follows from the fact that the polar line of the meeting point of the diagonals with respect to both the incircle and the circumcircle is the line passing through the meeting points of the extensions of opposite sides). Thus the points H, I, and M_A are collinear.

Let A_3 be the point opposite to A' on ω. It is known that the points A, A_3, and A_2 are collinear. Furthermore, the point M_A is the midpoint of A_2A'. Therefore, IM_A is a midline in the triangle $A_2A'A_3$. this yields that $HI \parallel AA_3$, i.e., the quadrilateral AA_3IH is a parallelogram, and so $r = A_3I = AH = 2OM_A$ (this quadrilateral is non-degenerate, otherwise $\triangle ABC$ would be isosceles). Therefore, M_AO is a midline in the triangle $IA'A_2$, since it passes through the midpoint of $A'A_2$ and is parallel to IA'; moreover, since $M_AO = r/2 = IA'/2$, we conclude that O is the midpoint of IA_2.

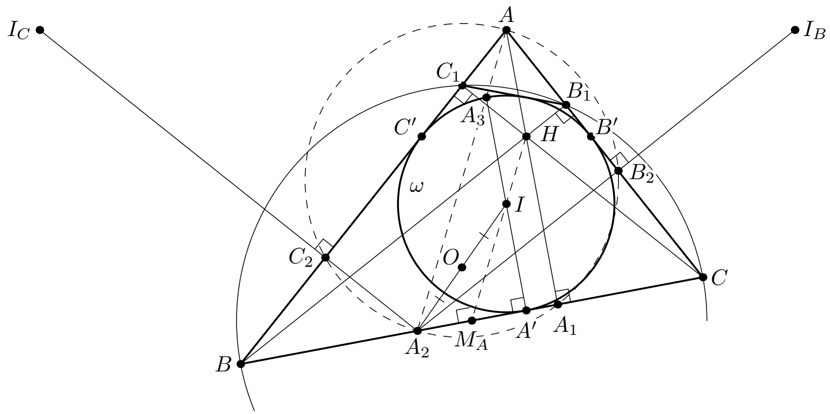

Fig. 10.4.1

Thus, the point A_2 is symmetric to I in O. Finally, the lines I_CC_2 and IC', as well as the lines I_BB_2 and IB', are symmetric in O; therefore, both these lines pass through A_2. Hence $\angle A_2B_2B = \angle A_2C_2C = 90^\circ = \angle AA_1B$. This means that all five points A, A_2, B_2, C_2, and A_1 lie on a circle with diameter AA_2.

Second solution. We present a different proof of the fact that $OM_A = r/2$; after that, one may finish the solution as shown above. The notation from the previous solution is still in force. Additionally, we denote by T the tangency point of the excircle ω_A with AB.

By the problem condition, the circle ω_A is an excircle of $\triangle AB_1C_1$. Therefore, the similarity transform mapping $\triangle AB_1C_1$ to $\triangle ABC$ sends ω to ω_A. Thus, $AB'/AT = \cos\angle A$. This means that $TB' \perp AC$, and hence TB' passes through I. Since $\angle TIC' = \angle A = \angle COM_A$, the right triangles $TC'I$ and COM_A are similar (Fig. 10.4.2). Moreover, we have $TC' = TB + BC' = BA_2 + CA_2 = BC = 2CM_A$, so the similarity ratio of these two triangles is 2; thus $OM_A = IC'/2 = r/2$.

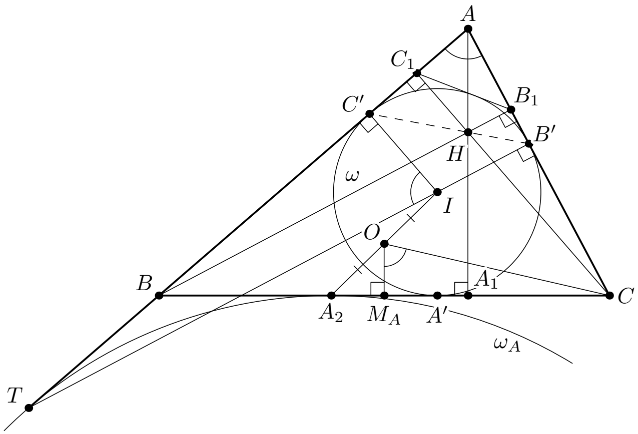

Fig. 10.4.2

Remark 1. In any triangle ABC, the circle $A_2B_2C_2$ is a pedal circle of a point K symmetric to I in O. This circle is also a pedal circle of a point K' isogonally conjugate to K. Thus, A_1 lies on the circle $A_2B_2C_2$ if and only if K' lies on AA_1, i.e., if either K lies on AO or $K' = A$. In the first case the triangle has to be isosceles, and in the second case we have $A_2 = K$.

Remark 2. One can also show that under the problem conditions, the orthocenter lies on the line $B'C'$ (Fig. 10.4.2), and the excircle tangent to the side BC is orthogonal to the circumcircle.

XI Geometrical Olympiad in honour of I.F.Sharygin
Final round. Grade 10. Second day

Ratmino, 2015, July 31.

5. (D. Shvetsov) Let BM be a median of right-angled nonisosceles triangle ABC ($\angle B = 90^\circ$), and H_a, H_c be the orthocenters of triangles ABM, CBM respectively. Lines AH_c and CH_a meet at point K. Prove that $\angle MBK = 90^\circ$.

   Solution. Since the lines AH_a and CH_c are perpendicular to BM, the quadrilateral AH_cCH_a is a trapezoid, and K is the common point of its lateral sidelines. Moreover, since the triangles ABM and CBM are isosceles, we have $H_aA = H_aB$ and $H_cC = H_cB$. Therefore, $KC/KH_a = CH_c/AH_a = BH_c/BH_a$, i.e. $KB \parallel CH_c \perp BM$ (Fig. 10.5).

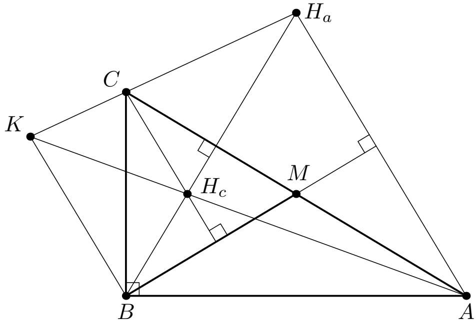

   Fig. 10.5

6. (A. Sokolov) Let H and O be the orthocenter and the circumcenter of triangle ABC. The circumcircle of triangle AOH meets the perpendicular bisector to BC at point A_1. Points B_1 and C_1 are defined similarly. Prove that lines AA_1, BB_1 and CC_1 concur.

   Solution. We will make use of the following fact.

   Lemma. Assume that a line ℓ passes through the orthocenter of a triangle; then the reflections of ℓ about the sidelines are concurrent.

   Proof. Let H be the orthocenter of a given triangle ABC. Then the points H_a, H_b, and H_c symmetric to H about BC, CA, and AB, respectively, lie on the circumcircle of the triangle. Furthermore, the angle subtended by the arc H_aH_b is equal to $2\angle C$, i.e., it is equal to the reflections of ℓ about BC and CA. Thus these reflections meet on the circumcircle. Clearly, the third line also passes through the same point. This finishes the proof of the lemma.

   Back to the problem. Consider the triangle $A'B'C'$ formed by the reflections of O about the sides of the triangle ABC. Its vertices are the circumcenters of the triangles HBC, HCA, and HAB; thus its sides are the perpendicular bisectors of HA, HB, and HC. Therefore, the sides of $\triangle A'B'C'$ are parallel to those of $\triangle ABC$, and O is its orthocenter (Fig. 10.6).

Now, the sides AH and A_1O of the cyclic quadrilateral $AHOA_1$ are parallel, so the lines AA_1 and OH are symmetric about the perpendicular bisector of AH, i.e., about $B'C'$. The similar statement holds for the lines BB_1 and CC_1. Hence these lines are concurrent due to the lemma applied to the triangle $A'B'C'$.

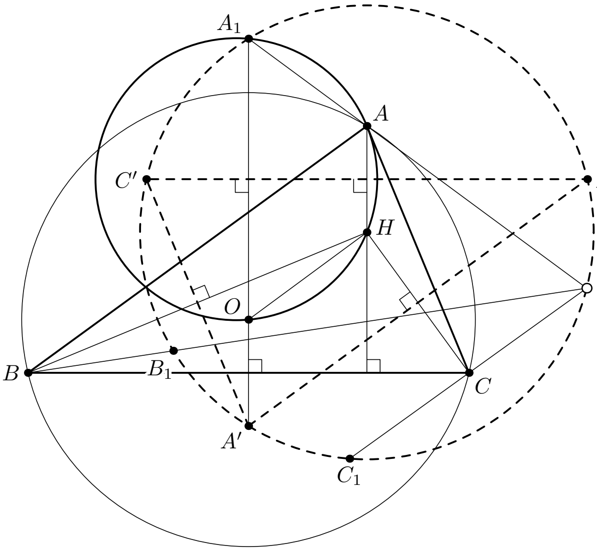

Fig. 10.6

7. (I.I.Bogdanov) Let $SABCD$ be an inscribed pyramid, and AA_1, BB_1, CC_1, DD_1 be the perpendiculars from A, B, C, D to lines SC, SD, SA, SB respectively. Points S, A_1, B_1, C_1, D_1 are distinct and lie on a sphere. Prove that points A_1, B_1, C_1 and D_1 are complanar.

   Solution. Since AA_1 and CC_1 are altitudes of the triangle SAC, the points A, C, A_1, and C_1 are concyclic, i.e., $SC \cdot SA_1 = SA \cdot SC_1$. Therefore, there exists an inversion centered at S interchanging A_1 with C and C_1 with A. Since $SB \cdot SD_1 = SD \cdot SB_1$, this inversion maps B_1 and D_1 to some points B_2 and D_2 on the rays SD and SB, respectively, such that $B_2D_2 \parallel BD$.

   On the other hand, the points A, C, B_2, and D_2 must be coplanar (as they are the images of the points lying on a sphere passing through S). However, if the line B_2D_2 does not lie in the plane $ABCD$, then the lines B_2D_2 and AC are skew. Thus, we are left with the only option that $B_2 = D$ and $D_2 = B$. Therefore, the points A_1, B_1, C_1, and D_1 lie in a plane which is the image of the sphere $SABCD$.

8. (M. Artemyev) Does there exist a rectangle which can be divided into a

regular hexagon with sidelength 1 and several equal right-angled triangles with legs 1 and $\sqrt{3}$?

Answer. No.

Solution. Suppose that such partition of some rectangle exists. Note that the area of each triangle in the partition is $S = \sqrt{3}/2$, and the area of the hexagon is equal to $3S$. Each side of the rectangle is partitioned into segments with lengths 1, 2, and $\sqrt{3}$, i.e., the lengths of these sides have the form $a+b\sqrt{3}$ and $c + d\sqrt{3}$ with nonnegative integers a, b, c, and d. Thus, the area of the rectangle equals

$$(a + b\sqrt{3})(c + d\sqrt{3}) = (ac + 3bd) + (ad + bc)\sqrt{3}.$$

On the other hand, this area is a multiple of S, therefore $ac + 3bd = 0$, i.e., $ac = 0$ and $bd = 0$.

This yields that one of these sides (say, vertical) has an integer length, while the other one (say, horizontal) has a length which is a multiple of $\sqrt{3}$. Thus the area of the rectangle is a multiple of $2S$. Since the area of the hexagon equals $3S$, the number of the triangles in the partition is odd. Now we prove that this is impossible.

Each (non-extendable) segment in the partition is covered by the segments of integer lengths and sides of length $\sqrt{3}$ on both sides. Thus the number of the segments of length $\sqrt{3}$ adjoining the segment is even. Next, none of the vertical sides of the rectangle adjoins such segments, while the horizontal sides of the rectangle are partitioned into such segments, and hence they also adjoin an even number of segments of length $\sqrt{3}$ in total. Thus the total number of sides of length $\sqrt{3}$ is even; but any triangle in the partition contains exactly one such side. This is a contradiction.