XIV Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. First day. 8 grade

1. (M.Volchkevich) The incircle of right-angled triangle ABC ($\angle C = 90^\circ$) touches BC at point K. Prove that the chord of the incicle cutting by the line AK is twice as large than the distance from C to this line.

   Solution. Let I be the incenter of ABC, and P, Q be the projections of I, C respectively to AK (fig.8.1). Since $\angle IKC = 90^\circ$, $\angle ICK = 45^\circ$, we obtain that IKC is an isosceles triangle, i.e. $IK = KC$. Also $\angle IKP = \angle KCQ$ because the corresponding sides of these angles are perpendicular. Therefore triangles IKP and KCQ are congruent, i.e. $KP = CQ$. Since P is the midpoint of the chord which is cut by AK, we obtain the required assertion.

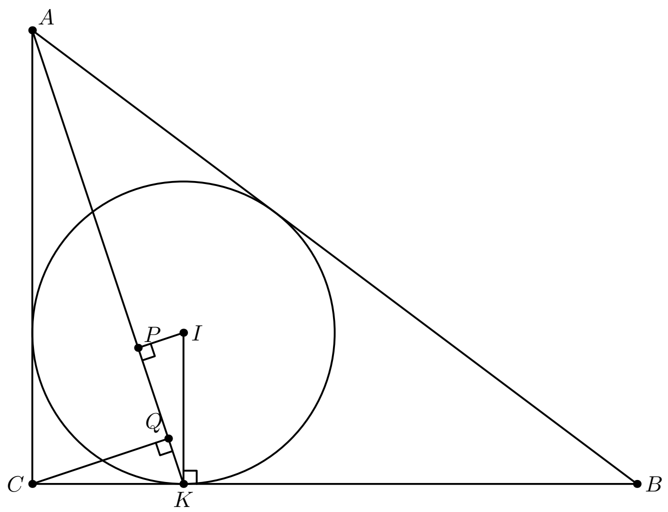

Fig. 8.1

2. (N.Moskvitin) A rectangle $ABCD$ and its circumcircle are given. Let E be an arbitrary point lying on the minor arc BC. The tangent to the circle at B meets CE at point G. The segments AE and BD meet at point K. Prove that GK and AD are perpendicular.

   Solution. Since $\angle DBG = \angle AEC = 90^\circ$, we obtain that $BGEK$ is cyclic (fig.8.2). Hence $\angle BGK = \angle BEA = \angle DBC$ and $GK \perp BC$, which is equivalent to the required assertion.

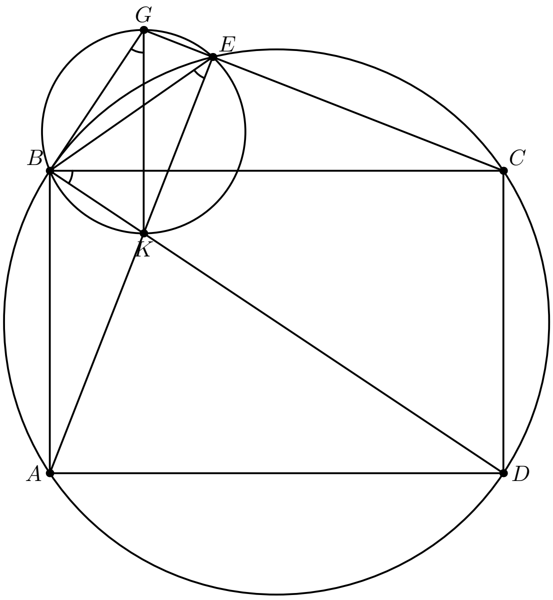

Fig. 8.2

3. (G.Feldman) Let ABC be a triangle with $\angle A = 60^\circ$, and AA', BB', CC' be its bisectors. Prove that $\angle B'A'C' \le 60^\circ$.

   Solution. If ABC is regular then the assertion is evident, thus we can suppose that $AC > AB$. Let I be the incenter. Then $\angle BIC = 120^\circ$, therefore $AB'IC'$ is cyclic, and since AI is the bisector, we obtain that $B'I = C'I$. Let $\angle ACB = 2\gamma$, then $\gamma < 30^\circ$ and $IA' = \frac{r}{\sin \angle AA'B} = \frac{r}{\sin(2\gamma+30^\circ)} > \frac{r}{\sin(\gamma+60^\circ)} = \frac{r}{\sin \angle CC'B} = IC'$. Hence A' lies outside the circle with center I and radius IC' (fig.8.3), i.e. $\angle B'A'C' < 60^\circ$.

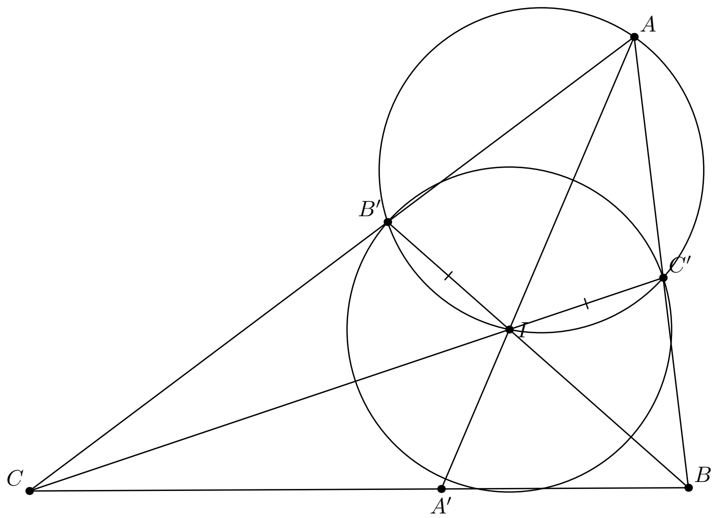

Fig. 8.3

4. (M.Saghafian) Find all sets of six points in the plane, no three collinear, such that if we divide them arbitrarily into two sets of three points, then two obtained triangles are equal.

Answer. Two regular triangles with common circumcircle.

First solution. Let D be the multiset of 15 distances between points $A_1, \ldots, A_6$ (if there are n congruent segments then the multiset contains n corresponding numbers), and let D_i be the multiset of 5 distances from A_i to the remaining points. Consider the multiset of 30 sidelengths of triangles having A_i as one of vertices. This multiset contains four times each number from D_i and one time each number from $D \setminus D_i$. By the assumption, the sidelengths of triangles not having A_i as a vertex form the same multiset of 30 numbers, i.e. this multiset contains three times each number from $D \setminus D_i$. Therefore $D = 3D_i$ and all D_i coincide.

Introduce an arbitrary Бartesian coordinate system, and let M be the point such that each coordinate of M is the average of the corresponding coordinates of A_i. Let X be an arbitrary point, and $x, m, a_1, \ldots, a_6$ be first coordinates of $X, M, A_1, \ldots, A_6$ respectively. Then we have $(x - a_1)^2 + \cdots + (x - a_6)^2 = ((x - m) + (m - a_1))^2 + \cdots + ((x - m) + (m - a_6))^2 = 6(x - m)^2 + (m - a_1)^2 + \cdots + (m - a_6)^2$. Using the similar equality for the second coordinates and Pythagorean theorem we obtain

$$XA_1^2 + \cdots + XA_6^2 = 6XM^2 + MA_1^2 + \cdots + MA_6^2$$

(this equality is a partial case of the Leibnitz theorem). Substituting $A_1, \ldots, A_6$ for X we obtain that $MA_1 = \cdots = MA_6$, i.e. all given points are concyclic. Suppose that they form a cyclic hexagon $A_1 \ldots A_6$. Let A_1A_2 be its minor side. Since all multisets D_i are equal, we obtain that $A_1A_2 = A_3A_4 = A_5A_6$. Similarly $A_2A_3 = A_4A_5 = A_6A_1$. It is easy to see that these conditions are sufficient.

Second solution. Let A_1, A_2,..., A_6 be the given points. Here triangle, segment and length mean a triangle with vertices A_i, a segment with endpoints A_i and a length of such segment respectively. Let us prove several lemmas.

1) For each length x one of the following assertions is true:

(A) there exists a regular triangle with sidelength x;

(B) there exist three segments with sidelength x having six different endpoints.

In fact, let $A_1A_2 = A_3A_4 = x$. Since $\triangle A_1A_2A_4 = \triangle A_3A_5A_6$, there is a side with length x in the $\triangle A_3A_5A_6$. If this is A_5A_6, we obtain (B), else we have two adjacent segments with length x. Let (after renumeration) $A_1A_2 = A_2A_3 = x$ and $A_4A_5 = A_5A_6 = x$. Since $\triangle A_2A_3A_5 = \triangle A_1A_4A_6$, there is a side with length x in the $\triangle A_1A_4A_6$. If this is A_4A_6, we have (A), else we have a broken line with five links of length x. Its extreme and medial links satisfy (B).

2) Let x be the maximal length. Then (A) is not correct, and thus (B) is true.

In fact if $\triangle A_1A_2A_3$ is regular and x is its sidelength, then the vertices of a congruent $\triangle A_4A_5A_6$ lie inside the corresponding Reuleaux triangle which is impossible.

3) Two segments with maximal length x intersect.

In fact let $A_1A_2 = x$, draw two lines to A_1 and A_2 perpendicular to A_1A_2. The remaining points lie inside the strip between this lines. Constructing the similar strip for $A_3A_4 = x$, we obtain that A_1A_2 and A_3A_4 are the altitudes of a rhombus, joining inner points of its sides. It is clear that such altitudes intersect.

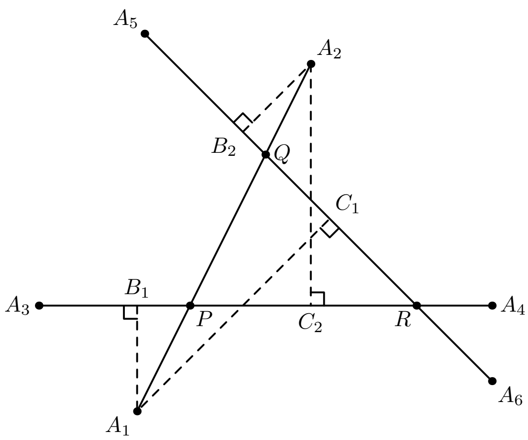

Fig. 8.4

4) Let segments A_1A_2, A_3A_4 and A_5A_6 intersect at three points (fig.8.4). Then the perpendiculars A_1B_1 and A_2B_2 are equal as altitudes of congruent triangles $A_1A_3A_4$ and $A_2A_5A_6$. Similarly the perpendiculars A_1C_1 and A_2C_2

are equal. Therefore $\dfrac{A_1P}{PA_2} = \dfrac{A_1B_1}{A_2C_2} = \dfrac{A_2B_2}{A_1C_1} = \dfrac{A_2Q}{QA_1}$ and $A_1P = QA_2$. This yields that $\triangle A_1PB_1 = \triangle A_2QB_2$, i.e. $\angle P = \angle Q$. Similarly we obtain that $\triangle PQR$ is regular, and thus $A_1P = QA_2 = A_3P = RA_4 = A_6R = QA_5$. It is easy to see that the obtained configuration satisfies the condition.

If three maximal segments concur then we similarly obtain that the angles between them are equal 60° and their common point bisects them.

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5. (S.Sevastianov) The side AB of a square $ABCD$ is a base of an isosceles triangle ABE ($AE = BE$) lying outside the square. Let M be the midpoint of AE, O be the common point of AC and BD, and K be the common point of ED and OM. Prove that $EK = KO$.

   Solution. Since OM is a medial line of triangle ACE, $OM \parallel EC$, therefore $\angle KOE = \angle OEC$ (fig.8.5). But it is clear that EO bisects angle CED. Thus $\angle EOK = \angle OEK$ and OKE is an isosceles triangle.

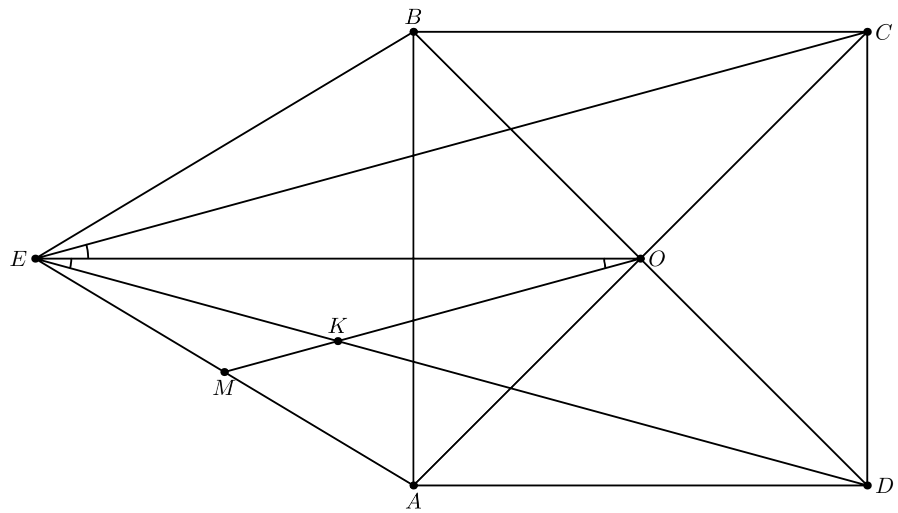

Fig. 8.5

6. (D.Shnol) The corresponding angles of quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ are equal. Also $AB = A_1B_1$, $AC = A_1C_1$, $BD = B_1D_1$. Are the quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ congruent?

   Answer. No.

   Solution. Let $A = A_1$, $B = B_1$, AXB be an isosceles triangle, AA', BB' be its altitudes. Let C, C_1 lie on BX and D, D_1 lie on AX in such a way that $CA' = C_1A' = DB' = D_1B'$. Then $AC = AC_1 = BD = BD_1$ and two isosceles trapezoids $ABCD$, $A_1B_1C_1D_1$ satisfy all conditions but are not congruent (fig.8.6).

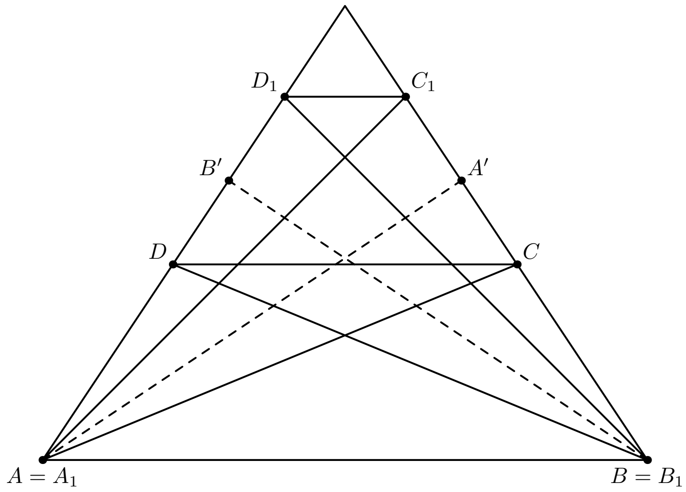

Fig. 8.6

7. (F.Nilov) Let ω_1, ω_2 be two circles centered at O_1, O_2 and lying each outside the other. Points C_1, C_2 lie on these circles in the same semiplane with respect to O_1O_2. The ray O_1C_1 meets ω_2 at points A_2, B_2, and the ray O_2C_2 meets ω_1 at points A_1, B_1. Prove that $\angle A_1O_1B_1 = \angle A_2O_2B_2$ if and only if $C_1C_2 \parallel O_1O_2$.

   First solution. Let R_1, R_2 be the radii of the circles, M_1, M_2 be the midpoints of A_1B_1, A_2B_2 respectively, and H_1, H_2 be the projections of C_1, C_2 to O_1O_2. The equality $\angle A_1O_1B_1 = \angle A_2O_2B_2$ is equivalent to $O_1M_1/R_1 = O_2M_2/R_2$. Since the triangle $O_1O_2M_2$ is similar to $O_1C_1H_1$, we have $O_2M_2/R_2 = (C_1H_1 \cdot O_1O_2)/(R_1R_2)$ (fig.8.7). Similarly $O_1M_1/R_1 = (C_2H_2 \cdot O_1O_2)/(R_1R_2)$. Therefore the equality $O_1M_1/R_1 = O_2M_2/R_2$ is equivalent to $C_1H_1 = C_2H_2$, which is equivalent to $C_1C_2 \parallel O_1O_2$.

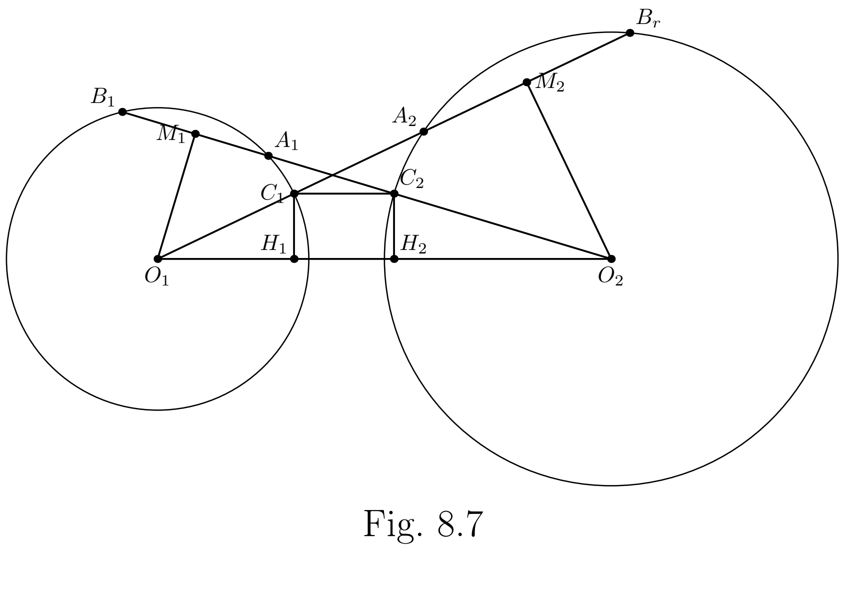

Fig. 8.7

Second solution. The equality $\angle A_1O_1B_1 = \angle A_2O_2B_2$ is equivalent to $\angle O_1A_1O_2 = \angle O_1A_2O_2$, i.e. $O_1A_1A_2O_2$ is cyclic. Let us prove that this is equivalent to $C_1C_2 \parallel O_1O_2$.

If $O_1A_1A_2O_2$ is cyclic then $\angle A_1O_1C_1 = \angle A_2O_2C_2$, $\angle O_1C_1A_1 = \angle O_2C_2A_2$ and $C_1A_1A_2C_2$ is cyclic. Therefore O_1O_2 and C_1C_2 are antiparallel to A_1A_2 with respect to O_1A_2 and O_2A_1. Hence these lines are parallel.

If $C_1C_2 \parallel O_1O_2$ then consider a common point X of ray O_1C_1 and circle $A_1O_1O_2$. Since $A_1C_1C_2X$ is cyclic we have $\angle A_1O_1X = \angle XO_2A_1$ and $\angle O_1C_1A_1 = \angle O_2C_2X$. Thus $\angle O_2XC_2 = \angle O_1A_1C_1 = \angle O_1C_1A_1 = \angle O_2C_2X$, i.e. $O_X = O_2C_2$ and X coincides with A_2.

8. (I.Kukharchuk) Let I be the incenter of triangle, and D be an arbitrary point of side BC. The perpendicular bisector to AD meets BI and CI at points F and E respectively. Find the locus of orthocenters of triangles EIF.

   Answer. The segment of line BC between its common points with two lines passing through I and parallel to AB, AC, probably without one or two points.

   Solution. Let G, H be the orthocenters of triangles DEF, IEF respectively. Since the triangles DEF and AEF are symmetric with respect to EF, we obtain that G is the reflection of the orthocenter of AEF and thus G lies on the circumcircle of this triangle.

   The common point E of the perpendicular bisector to AD and the bisectrix of angle C lies on the circumcircle of ACD. Hence $\angle AEF = \angle AED/2 = 90^\circ - \angle C/2 = \angle A/2 + \angle B/2 = \angle AIF$ (because AIF is an external angle of triangle AIB), i.e. I lies on the circle AEF. Then, since $AEDC$ and $AEIG$ are cyclic, we obtain that $IG \parallel CD$.

   Since $\angle EHF = 180^\circ - \angle EIF = \angle EAF = \angle EDF$, the points E, F, D, H are concyclic, therefore $IH = DG$. Also it is clear that $DG \parallel IH$. Thus $IGDH$ is a parallelogram and H lies on BC (fig.8.8). If for example D coincides with C then DG coincides with AC and $IH \parallel AC$. If BC is the smallest side of the triangle then all points of the obtained segment lie on the required locus. If for example $BC \geq AB$ then the reflection of A about the bisector of angle B lies on the segment BC. When D coincides with this point the perpendicular bisector to AD coincides with BI and the point F is not defined. Hence the corresponding point H has to be eliminated from the locus.

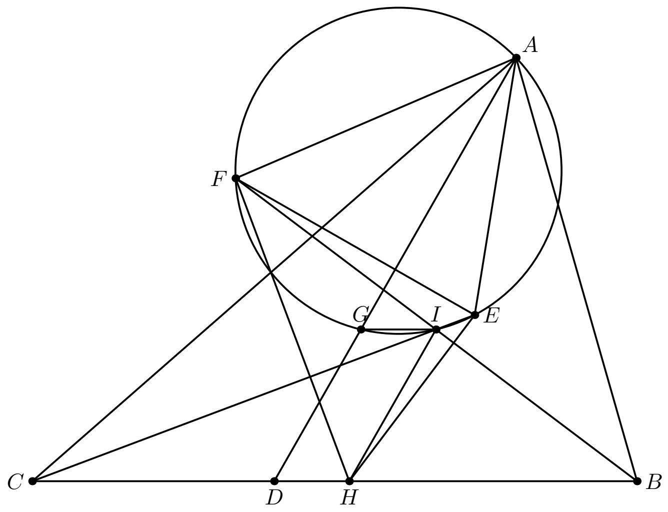

Fig. 8.8

Note. We can prove that H lies on BC in another way. The projections of A to the bisectors of angles B and C lie on the medial line of the triangle. The midpoint of AD which is the projection of A to EF also lies on this medial line. Hence the medial line is the Simson line of A with respect to triangle IEF, and the homothetic line BC passes through the orthocenter of this triangle.

XIV Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. First day. 9 grade

1. (M.Etesamifard) Let M be the midpoint of AB in a right-angled triangle ABC with $\angle C = 90°$. A circle passing through C and M intersects the segments BC and AC at P and Q, respectively. Let c_1, c_2 be circles with centers P, Q and radii BP, CQ, respectively. Prove that c_1, c_2 and the circumcircle of ABC are concurrent.

   Solution. Let N be the second common point of circle MPQ with AB. Then $\angle QNA = \angle QPM = \angle ACM = \angle CAM$ (fig.9.1). Therefore $QA = QN$ and N lies on c_2. Similarly N lies on c_1. Now if D is the second common point of c_1 and c_2 then $\angle ADB = \angle ADN + \angle NDB = (\angle AQN + \angle NPB)/2 = 90°$, i.e. D lies on the circumcircle of ABC.

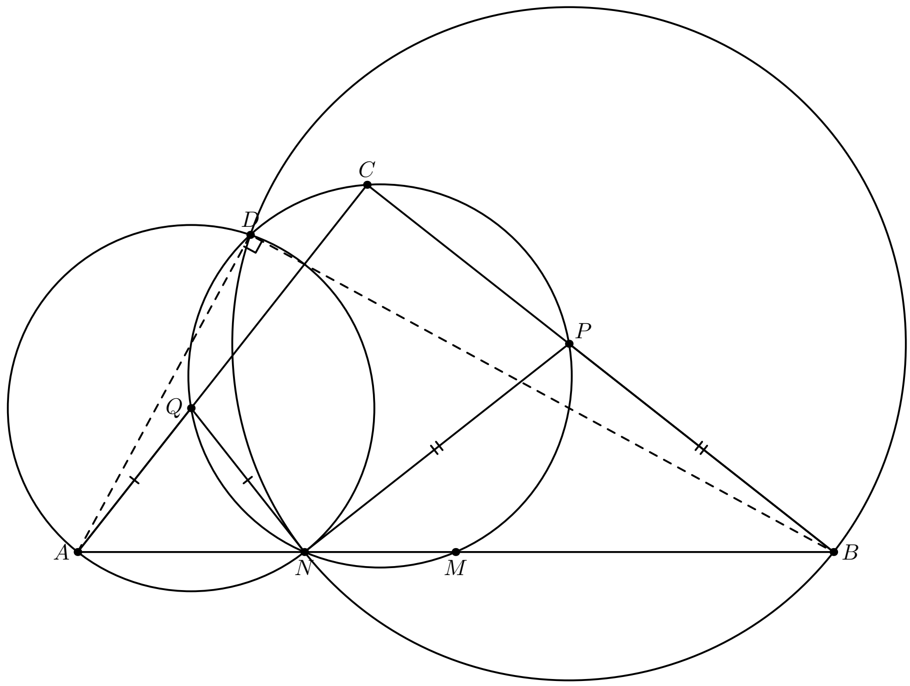

Fig. 9.1

2. (G.Naumenko) A triangle ABC is given. A circle γ centered at A meets segments AB and AC. The common chord of γ and the circumcircle of ABC meets AB and AC at points X and Y respectively. The segments CX and BY meet γ at points S and T respectively. The circumcircles of triangles ACT and BAS meet at points A and P. Prove that CX, BY and AP concur.

   Solution. Let U be the second common point of BY and γ. Since TU, AC and the common chord of circles ABC and γ meet at Y, we have $AY \cdot CY =$

$TY \cdot UY$, i.e. A, U, C, T are concyclic (fig.9.2). Similarly A, B, S and the second common point of CX with γ are concyclic. Therefore CX, BY and AP concur as the radical axes of circles γ, ACT and BAS.

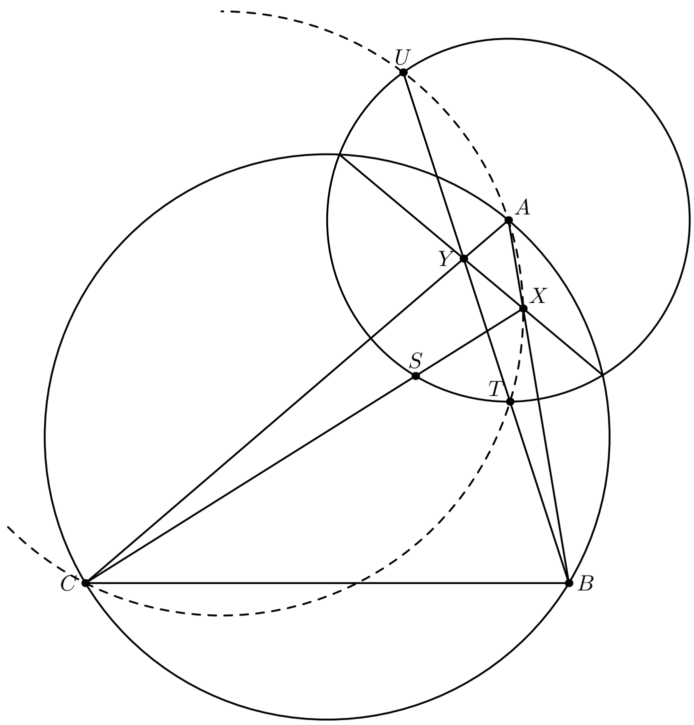

Fig. 9.2

3. (N.Beluhov) The vertices of triangle DEF lie on the different sides of triangle ABC. The tangents from the incenter of DEF to the excircles of ABC are equal. Prove that $4S_{DEF} \geq S_{ABC}$.

   Solution. Let A_0, B_0, C_0 be the midpoints of BC, CA, AB, and U, V be the tangency points of AB with the excircles touching the sides AC and BC respectively. Since $AV = BU = p$ (semiperimeter of ABC), the tangents from C_0 to these two excircles are equal. Furthemore the centers of these circles lie on the external bisector of angle C perpendicular to the bisector of angle $A_0C_0B_0$, hence this bisector is the radical axis of two excircles. Similarly the bisectors of angles $C_0A_0B_0$ and $B_0A_0C_0$ are the radical axes of two remaining pairs of excircles, thus the incenters of triangles DEF and $A_0B_0C_0$ coincide. Suppose that D lies on the segment CA_0. Now if the inradius r' of DEF is greater than the inradius r of $A_0B_0C_0$ then F lies on the segment BC_0, and thus E lies on AB_0. Furthermore if $r' < r$ then E lies on AB_0, and thus F lies on BC_0. Hence the distance from F to ED is not less than the distance from C_0 to this line, i.e. $S_{DEF} \geq S_{C_0DE}$. Similarly $S_{C_0DE} \geq S_{B_0C_0D} = S_{A_0B_0C_0} = S_{ABC}/4$.

4. (A.Mudgal, India) Let BC be a fixed chord of a given circle ω. Let A be a variable point on the major arc BC of ω. Let H be the orthocenter of triangle ABC. Points D and E lying on lines AB and AC respectively are such that H is the midpoint of segment XY . Let O_A be the circumcenter of triangle AXY. Prove that all points O_A lie on a fixed circle.

   Solution Denote by α the constant angle $90° - \angle BAC$. Let P, Q be the midpoints of AD, AE, and R, S be points on BC such that $PR \perp AB$, $SQ \perp AC$ (fig. 9.4). Let us prove that R, S do not depend from A.

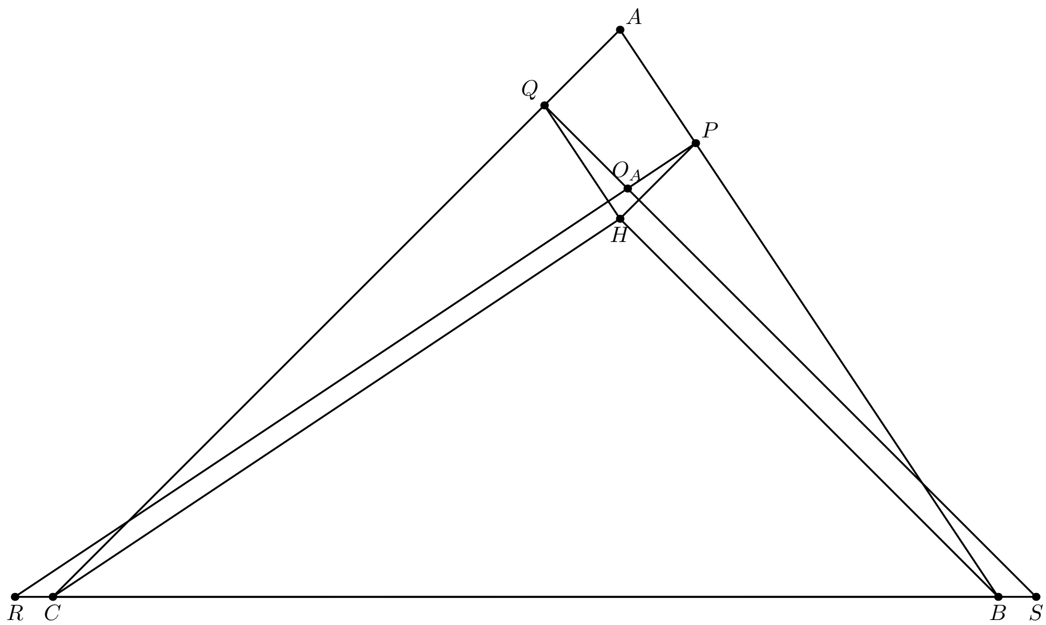

Fig. 9.4

   Note that $HQ \parallel AB$, i.e. $\angle CHQ = 90°$ and $\angle CQH = \angle CAB$. Furthermore H moves aloong the circle symmetric to ω with respect to BC. Since Q is the image of H in the spiral similarity with center C, rotation angle α and coefficient $1/\cos\alpha$, we obtain that Q also moves along some circle which we denote by ω_C.

   Let O be the center of ω. Since $\angle OCB = \alpha$, the center of ω_C lies on BC. Since $\angle CQS = 90°$, we obtain that S is opposite to C on ω_C. Therefore S does not depend from A. The proof for R is similar.

   Since O_A is the common point of PR and QS, and $\angle RO_AS = 90° + \alpha$, we obtain that O_A moves along the arc of the circle passing through R and S.

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Final round. Solutions. Second day. 9 grade

5. (D.Prokopenko) Let $ABCD$ be a cyclic quadrilateral, BL and CN be the bisectors of triangles ABD and ACD respectively. The circumcircles of triangles ABL and CDN meet at point P and Q. Prove that the line PQ passes through the midpoint of the arc AD not containing B.

   Solution. Let M be the midpoint of arc AD. Then BL and CN pass through M. Also since $\smile AM = \smile DM$, we have $\angle ALB = (\smile AB + \smile DM)/2 = \smile BAM/2 = \angle BCM$, and thus $BCNL$ is cyclic (fig.9.5). Therefore $ML \cdot MB = MN \cdot MC$, and M lies on the radical axis PQ of circles ABL and CDN.

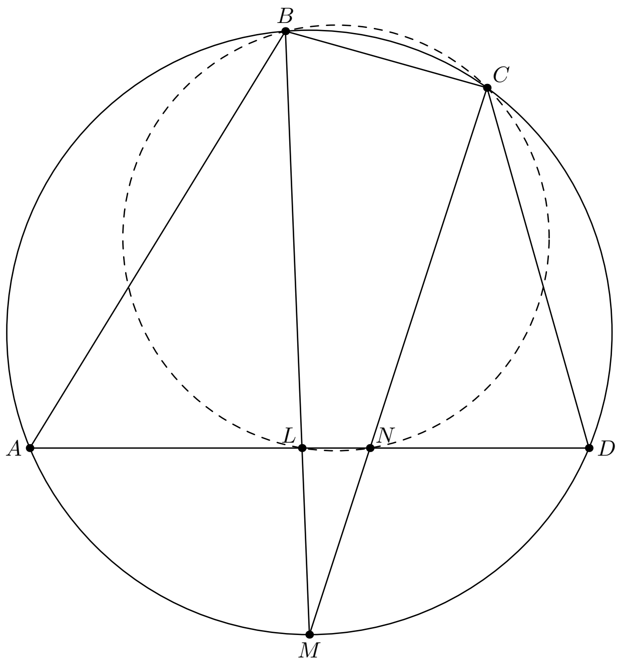


Fig. 9.5

6. (F.Ivlev) Let $ABCD$ be a circumscribed quadrilateral. Prove that the common point of its diagonals, the incenter of triangle ABC and the center of excircle of triangle CDA touching the side AC are collinear.

   First solution. Applying the three homothety centers to the incircle of $ABCD$, the incircle ω of triangle ABC and the excircle Ω of triangle ACD, we obtain that the common external tangents to ω and Ω meet on BD, and since AC is one of these two tangents, they meet at the common point of diagonals of $ABCD$. Thus this point lies on the centerline of ω and Ω.

Second solution. Let L be the common point of the diagonals of $ABCD$, I be its incenter, I_B be the incenter of triangle ABC, and I_D be the excenter of triangle ADC. Clearly I_B lies on the segment BI, and the ratio $BI_B : BI$ is equal to the ratio $r_B : r$ of the inradii of ABC and $ABCD$ respectively. Since $S_{ABCD} = (AB+BC+CD+DA)r/2$, $S_{ABC} = (AB+BC+CA)r_B/2$ and $S_{ABC} : S_{ABCD} = BL : BD$, we obtain that

$$\frac{I_BI}{I_BB} = \frac{DL(AB+BC+CA) - BL(AD+CD-AC)}{BL(AB+BC+CD+DA)}.$$

Similarly for the point I_D lying on the ray DI we have

$$\frac{I_DI}{I_DD} = \frac{DL(AB+BC+AB) - BL(AD+CD-AC)}{DL(AB+BC+CD+DA)}.$$

Applying Menelaos theorem to the triangle IBD, we obtain the required assertion (fig.9.6).

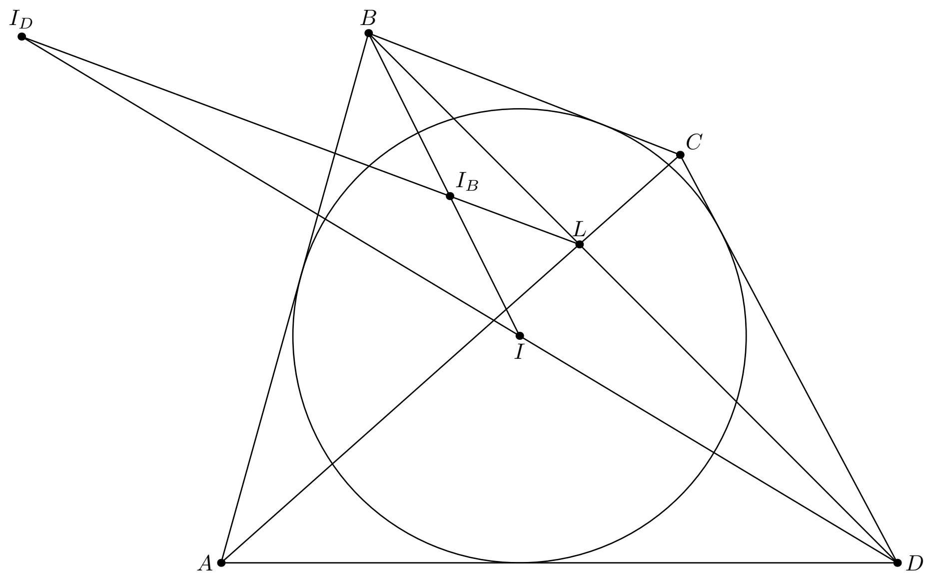


Fig. 9.6

7. (A.Kulikova) Let B_1, C_1 be the midpoints of sides AC, AB of a triangle ABC. The rays CC_1, BB_1 meet the tangents to the circumcircle at B and C at K and L respectively. Prove that $\angle BAK = \angle CAL$.

   Solution. Use **the isogonals theorem**.

   Let ℓ be a line passing through a point O. Let points A, A', B, B' be given and $X = AB \cap A'B'$, $X' = AB' \cap A'B$. Let OA and OA' be symmetric with respect to ℓ, OB and OB' be also symmetric with respect to ℓ. Then OX and OX' are symmetric with respect to ℓ,

Return to the problem. Let M be the centroid of ABC,and P be the common point of two tangents. Since AP is a symedian, the lines AP and AM are the isogonals with respect to angle BAC (fig.9.7). By the isogonals theorem, AK and AL are also isogonals.

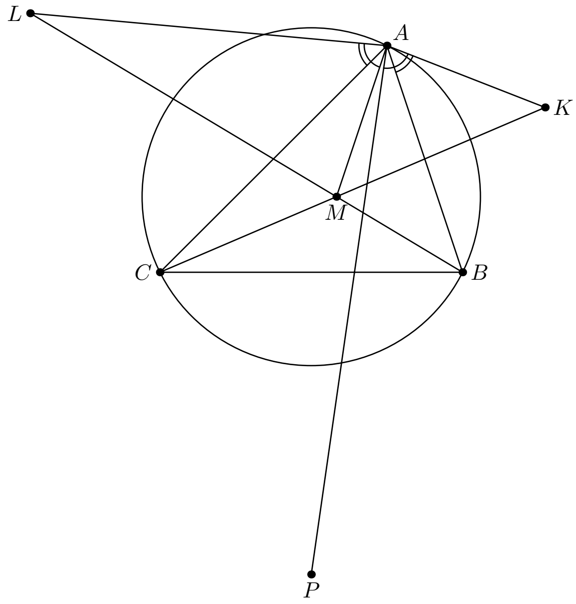
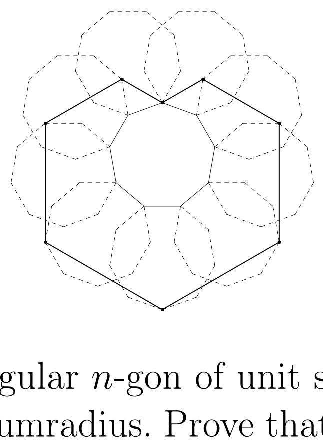


Fig. 9.7

8. (N.Beluhov) Consider a fixed regular n-gon of unit side. As a second regular n-gon of unit side rolls around the first one, one of its vertices successively pinpoints the vertices of a closed broken line κ as in the figure.



Let A be the area of a regular n-gon of unit side and let B be the area of a regular n-gon of unit circumradius. Prove that the area enclosed by κ equals $6A - 2B$.

Solution. Dissect the area enclosed by κ into triangles as in Fig.9.8.1.

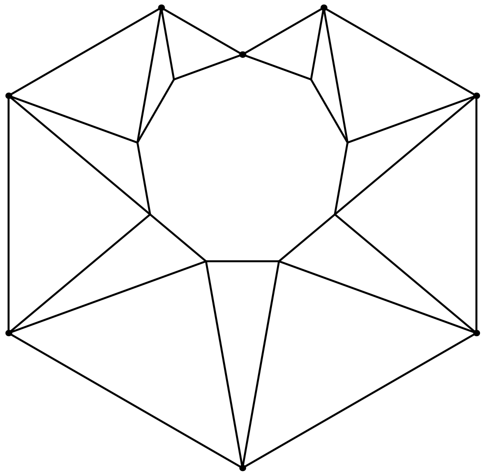


Fig.9.8.1

The triangles whose bases are sides $2, \ldots, n-1$ of a fixed regular n-gon come together to form a regular n-gon of unit side as in Figure 9.8.2.

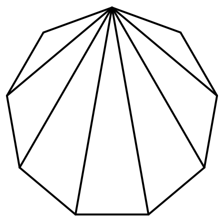


Fig.9.8.2

Dissect two regular n-gons of unit circumradius as in Figure 9.8.2, rearrange the resulting pieces into $n-1$ similar isosceles triangles with base angle $\frac{180°}{n}$ as in Figure 9.8.3, and adjoin the triangles thus obtained to the remaining triangles of Figure 9.8.1 as in Figure 9.8.4.

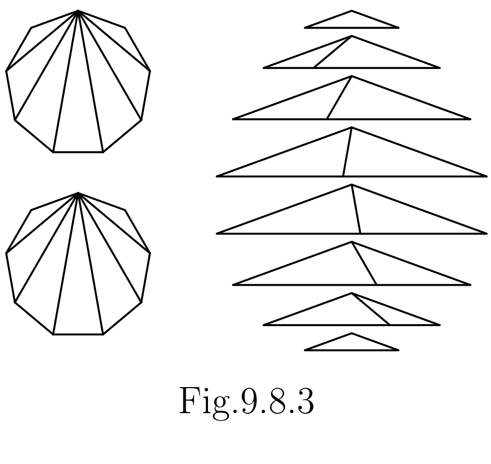


Fig.9.8.3

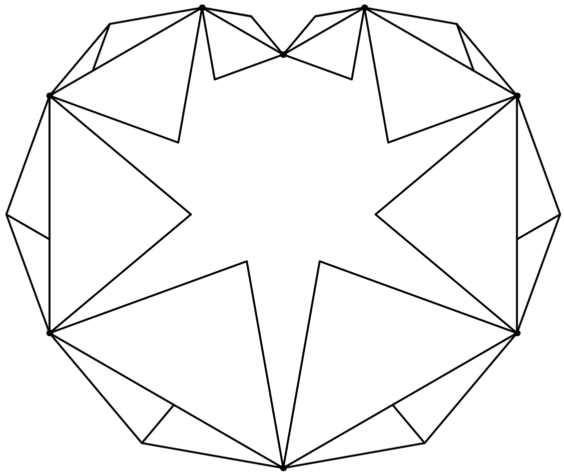


Fig.9.8.4

Dissect each one of the $n-1$ quadrilaterals in Figure 4 into two similar isosceles triangles with base angle $\frac{180°}{n}$ as in Figure 9.8.5.

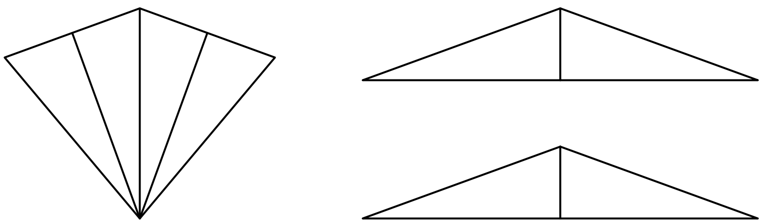


Fig.9.8.5

Lastly, dissect all $2n-2$ triangles thus obtained into four regular n-gons of unit side by the reverse of the process in Figure 9.8.3.

Eventually we adjoined two regular n-gons of unit circumradius to the area enclosed by κ and then dissected the resulting shape into six regular n-gons with unit side. This completes the solution.

XIV Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. First day. 10 grade

1. (D.Shvetsov) The altitudes AH, CH of an acute-angled triangle ABC meet the internal bisector of angle B an points L_1, P_1, and the external bisector of this angle at points L_2, P_2. Prove that the orthocenters of triangles HL_1P_1, HL_2P_2 and the vertex B are collinear.

   First solution. Note that HL_1P_1 and HL_2P_2 are isosceles triangles with angles at H equal to B and $\pi - B$ respectively. Let H_1, H_2 be the orthocenters of these triangles, and M_1, M_2 be the midpoints of L_1P_1, L_2P_2 respectively. Then the triangles HL_2P_2, $H_1L_1P_1$ are similar and H_2, H are their orthocenters, therefore $HH_1 : M_2B = HH_1 : HM_1 = H_2H : H_2M_2$, which is equal to the required assertion (fig.10.1).

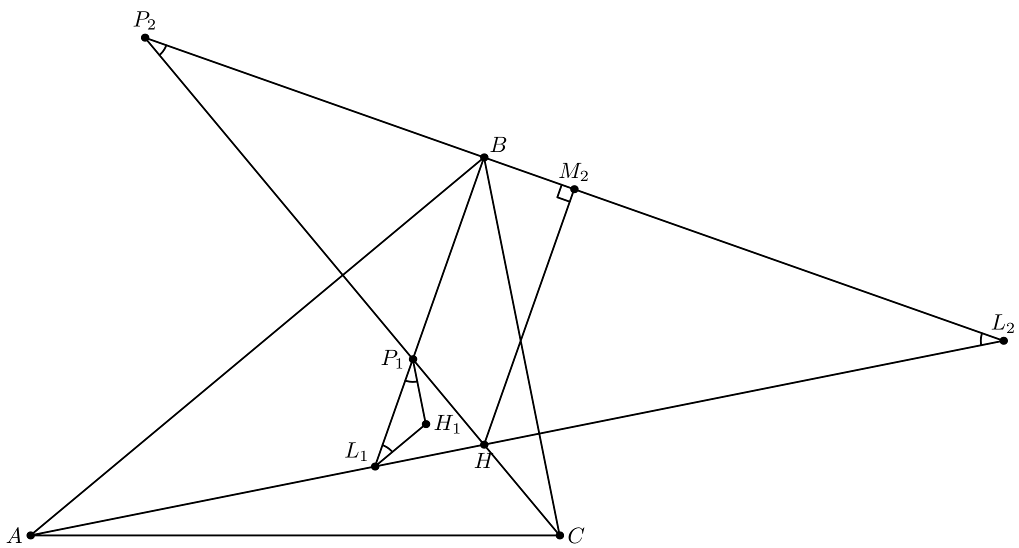


Fig. 10.1

   Second solution. Use next fact.

   The orthocenters of four triangles formed by four lines in general position are collinear (**the Aubert line**).

   In the given case the altitudes from A, C, the internal and the external bisectors of angle B form four four triangles, two of them are right-angled with right angle B. Thus B is also the orthocenter of these triangle, therefore B and the orthocenters of two remaining triangles are collinear.

2. (D.Krekov) A circle ω is inscribed into an angle with vertex C. An arbitrary circle passes through C, touches ω externally and meets the sides of the angle at points A and B. Prove that the perimeters of all triangles ABC are equal.

First solution. Let the length of the tangent from C to the given circle is 1. The inversion about the unit circle centered at C preserves the sides of the angle and the given circle, and maps A, B to points A', B' such that the triangle $A'B'C$ is circumscribed around the given circle. Now we have $AC = 1/A'C$, $BC = 1/B'C$, $AB = A'B'/(A'C \cdot B'C)$. Hence the perimeter of ABC is equal to

$$\frac{A'B' + A'C + B'C}{A'C \cdot B'C} = \frac{2p_{A'B'C} \sin\angle C}{2S_{A'B'C}} = \frac{\sin\angle C}{r_{A'B'C}}.$$

But the inradius of $A'B'C$ does not depend from A, B.

Second solution. Since ω is the semiexcircle of triangle ABC, the excenter ot these triangle coincide with the midpoint of the segment between the touching points of ω with the sidelines of the given angle, т.е this excenter is the same for all triangles. Therefore the touching points of the excircle with the sidelines do not depend on the triangle, hence its perimeter is also constant.

3. (F.Nilov) A cyclic n-gon is given. The midpoints of all its sides are concyclic. The sides of n-gon cut n arcs of this circle ling outside the n-gon. Prove that these arcs can be colored red and blue in such a way that the sum of red arcs is equal to the sum of blue arcs.

   Solution. Let M_1, M_2 be the midpoints of sides A_1A_2, A_2A_3 of polygon $A_1 \dots A_n$, O be the circumcenter of this polygon, and H_1, H_2 be the second common points of the sides with the circle passing through the midpoints. Then the sum of directed arcs $\smile M_1H_1 + \smile M_2H_2 = \smile M_1H_2 + \smile M_2H_1 = 2(\angle A_2M_2M_1 + \angle A_2M_1M_2) = 2(\angle OM_2M_1 + \angle OM_1M_2) = 2(\angle OA_2M_1 + \angle OA_2M_2)$ (the last equality holds because $OM_1A_2M_2$ is cyclic). Summing up such equalities we obtain that the directed sum of arcs M_iH_i is zero, therefore we can color the arcs in correspondence with their directions.

   Note. We can modify this argumentation as follows. The projections M_i of the circumcenter O to the sides are concyclic. Therefore the second common points H_i of the sides and these circles are the projections of some point H, and the rays A_iO and A_iH are symmetric with respect to the bisector of angle $A_{i-1}A_iA_{i+1}$. Now it is easy to see that the directed angle between M_1M_2 and H_1H_2 is equal to the directed angle HA_2O, and the sum of such angles is zero.

4. (N.Beluhov) We say that a finite set S of red and green points in the plane is *orderly* if there exists a triangle δ such that all points of one colour lie

strictly inside δ and all points of the other colour lie strictly outside of δ. Let A be a finite set of red and green points in the plane, in general position. Is it always true that if any 1000 points in A form an orderly set then A is also orderly?

Solution. At first let us consider a slightly different problem, in which "orderly" is replaced with "red-orderly": there exists a triangle δ such that all red points lie strictly inside δ and all green points lie strictly outside of δ.

Let A be a finite set of red and green points in the plane, in general position, and let P be the convex hull of some red points in A. Let also Q be a subset of the green points in A. How can we check if a triangle δ exists that separates P from Q?

Without loss of generality, the sides of δ are supporting lines for P.

Let c be some fixed circle. To each supporting line l of P assign the unique point $T(l)$ on c such that the tangent $t(T)$ to c at $T(l)$ is parallel to l and P lies on the same side from l as c does of $t(T)$.

Let X be any point in Q. Let $l_1(X)$ and $l_2(X)$ be the two supporting lines of P through X (if X lies inside P, then P cannot be separated from Q), and let $a(X)$ be the arc of c with endpoints $T(l_1(X))$ and $T(l_2(X))$.

If a separating triangle δ exists, then the three points on c assigned to its sides nail down all arcs $a(X)$ where X ranges over Q. The converse statement is slightly more subtle: if there exist three points on c that nail down all arcs $a(X)$, and are the vertices of an acute-angled triangle, then they give us a triangle δ that separates P from Q.

So we are looking for a system of arcs on c such that it is possible to nail down all its large subsystems by means of three points but this is impossible for the complete system. (It is not clear yet whether finding such a system or showing that none exist for some sense of "large" would solve the problem, but it should surely shed some light.)

We construct such a system as follows: we consider a large number of equal, equally spaced arcs set up in such a way that any point nails down nearly but not quite a third of them.

More precisely, let n be a positive integer and let T_1, T_2, $\ldots$, T_{3n+1} be the vertices of a regular $(3n+1)$-gon inscribed in c, with $T_{i+3n+1} \equiv T_i$ for all i. For all i let a_i be the open arc T_iT_{i+n}. Then any point on c nails down at most n arcs, so it is impossible to nail down all arcs by means of three points.

On the other hand, remove any one arc, say T_1T_{n+1}, and three midpoints of $T_{n+1}T_{n+2}$, $T_{2n+1}T_{2n+2}$, and $T_{3n+1}T_1$ do the job.

This provides a counterexample for the original problem. Consider a regular 3001-gon $Y_1Y_2\dots Y_{3001}$ inscribed in a circle k of center O, where $Y_{i+3001} \equiv Y_i$ for all i. For all i let X_i be the intersection of the tangents to k at Y_i and Y_{i+1000}. Slide each X_i very slightly towards O so that all points are in general position. Colour all X_i green and all Y_i red, and let A be the set of all X_i and Y_i.

Since the convex hull of the X_i contains the Y_i, the set A can be orderly only if it is red-orderly. However, by the previous discussion, it is not.

Remove any X_i or any Y_i. Again, by the previous discussion, A becomes red-orderly and, therefore, orderly.

Notes. We say that a finite set S of red and green points in the plane is *line-orderly* if there exists a line l such that all points of one colour lie strictly on one side of l and all points of the other colour lie strictly on the other side of l. Let A be a finite set of red and green points in the plane, in general position. Then A is line-orderly if and only if every four-point subset of A is line-orderly.

We say that a finite set S of red and green points in the plane is *circle-orderly* if there exists a circle c such that all points of one colour lie strictly inside c and all points of the other colour lie strictly outside of c. Let A be a finite set of red and green points in the plane, in general position. Then A is circle-orderly if and only if every five-point subset of A is circle-orderly.

XIV Geometrical Olympiad in honour of I.F.Sharygin
Final round. Solutions. Second day. 10 grade

5. (A.Polyanskii) Let w be the incircle of a triangle ABC. The line passing through the incenter I and parallel to BC meets w at points A_B an A_C (A_B lies in the same semiplane with respect to AI that B). The lines BA_B and CA_C meet at point A_1. The points B_1 and C_1 are defines similarly. Prove that AA_1, BB_1 and CC_1 concur.

   First solution. Since segments A_BA_C and BC are homothetic with respect to A_1, the line A_1I passes through the midpoint M of BC and $A_1I : A_1M = 2r : BC$. Hence the distance from A_1 to AC is equal to $r(BC - h_b)/(BC - 2r)$, where h_b is the length of the altitude from B. Similarly the distance from A_1 to AB is equal to $r(BC - h_c)/(BC - 2r)$. Therefore $\sin\angle A_1AC : \sin\angle A_1AB = (1 - \sin\angle C) : (1 - \sin\angle B)$. Using the similar equalities for B_1, C_1 and Ceva theorem we obtain the required assertion.

   Second solution. Since $\angle A_BIB = \angle IBC = \angle IBA = \angle C_BIB$, the points A_B and C_B are symmetric with respect to the bisector of angle B (fig.10.5).

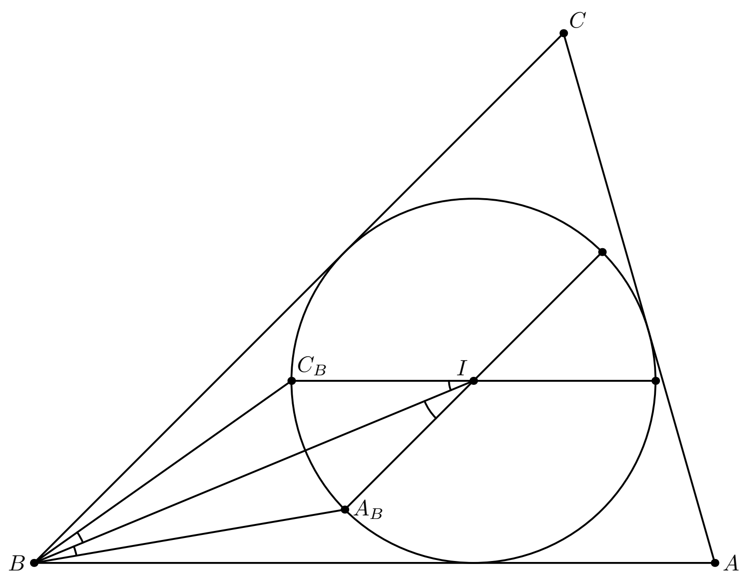


Fig. 10.5

By Ceva theorem

$$\frac{\sin\angle CAA_1}{\sin\angle BAA_1}\frac{\sin\angle ABA_B}{\sin\angle CBA_B}\frac{\sin\angle BCA_C}{\sin\angle ACA_C} = 1.$$

Multiplying this and two similar equalities we obtain the required assertion.

6. (M.Kungozhin) Let ω be the circumcircle of a triangle ABC, and KL be the diameter of ω passing through the midpoint M of AB (K and C lies on the different sides from AB). A circle passing through L и M meets segment CK at points P and Q (Q lies on the segment KP). Let LQ meet the circumcircle of triangle KMQ at point R. Prove that the quadrilateral $APBR$ is cyclic.

   Solution. Note that $\angle PML = \angle PQL = \angle KQR = \angle KMR$. Also $\angle PLM = \angle KQM = \angle KRM$, therefore the triangles PLM and KRM are similar, i.e. $PM \cdot RM = LM \cdot KM = AM^2$ (fig.10.6).

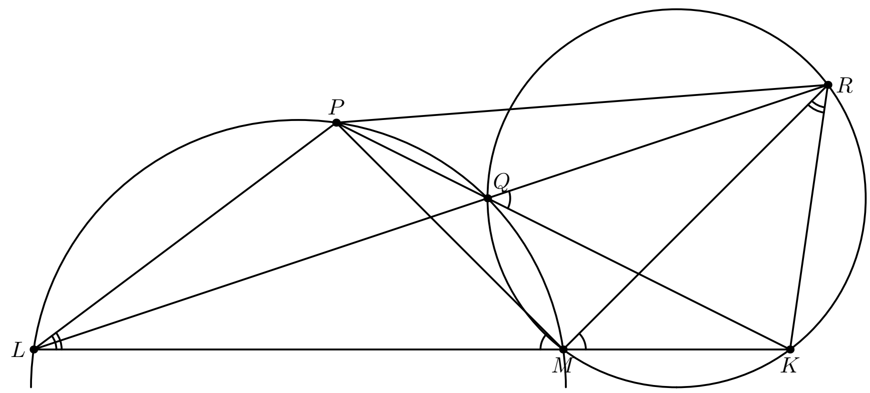


Fig. 10.6

   Let P' be the reflection of P about KL. The points A, B, P, P' are concyclic as the vertices of an isosceles trapezoid. Since P', M, R are collinear and $P'M \cdot RM = AM \cdot BM$, we obtain that R also lies on this circle.

7. (N.Beluhov) A convex quadrilateral $ABCD$ is circumscribed about a circle of radius r. What is the maximum possible value of $\frac{1}{AC^2} + \frac{1}{BD^2}$?

   First solution. Let $AC \cap BD = O$ and suppose without loss of generality that $\angle AOB \geq 90^\circ$. Construct E so that $BECD$ is a parallelogram (fig.10.7).

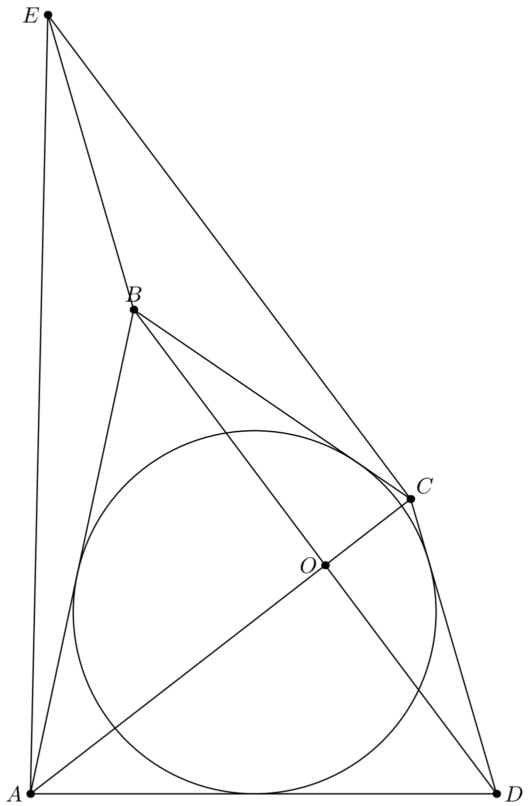


Fig. 10.7

We have

$$AC \cdot BD \geq 2S_{ABCD} = r \cdot P_{ABCD} = 2r \cdot (AB + CD).$$

Furthermore

$$AB + CD = AB + BE \geq AE$$

and (since $\angle ECA \geq 90^\circ$)

$$AE^2 \geq AC^2 + CE^2 = AC^2 + BD^2.$$

Hence

$$AC^2 \cdot BD^2 \geq 4r^2 \cdot (AC^2 + BD^2) \Rightarrow \frac{1}{AC^2} + \frac{1}{BD^2} \leq \frac{1}{4r^2}.$$

Equality is attained just for $AC \cdot BD = 2S_{ABCD} \Leftrightarrow AC \perp BD$ and $AB + BE = AE \Leftrightarrow AB \| CD$, that is, when $ABCD$ is a rhombus.

Second solution. We begin by altering $ABCD$ continuously so that its incircle remains the same but its diagonals become shorter.

Let the circle ω with center I be the incircle of $ABCD$. Fix ω, the line l determined by the points A and C, and the line m through B parallel to l. Let B vary along m. What happens to the length of AC?

Suppose the tangent n to ω parallel to both l and m and separating them, meets AB and BC at P and Q, and let a circle ω' of center I' and radius r' be the incircle of $\triangle PBQ$.

When B varies, the ratio of similitude of $\triangle PBQ$ and $\triangle ABC$ remains constant. This means that the ratio $PQ : AC$, the ratio $r' : r$, and r' all remain constant too.

Furthermore, PQ equals the common external tangent of ω' and ω. Since r' and r are constant, this common external tangent is shortest when II' is shortest, i.e., when $BI \perp l$. Since $PQ : AC$ is constant, AC is also shortest in this case.

Slide B along m until it reaches a position B_1 with $IB_1 \perp l$, then slide it along IB_1 towards I until it reaches a position B_2 such that the length of A_2C_2 equals the original length of AC. Do the same with D. Then $A_2B_2C_2D_2$ is circumscribed about ø, symmetric about B_2D_2, and satisfies $A_2C_2 = AC$ and $B_2D_2 \leq BD$.

Repeat this procedure with A_2 and C_2: the result is a rhombus $A_3B_3C_3D_3$ circumscribed about ω which satisfies $A_3C_3 \leq AC$ and $B_3D_3 \leq BD$. For a rhombus, though, we have

$$\frac{1}{A_3C_3^2} + \frac{1}{B_3D_3^2} = \frac{1}{4r^2}$$

and if $A_3B_3C_3D_3 \neq ABCD$ then at least one of inequalities $A_3C_3 \leq AC$ and $B_3D_3 \leq BD$ is strict.

8. (A.Zaslavsky) Two triangles ABC and $A'B'C'$ are given. The lines AB and $A'B'$ meet at point C_1, and the lines parallel to them and passing through C and C' respectively meet at point C_2. The points A_1, A_2, B_1, B_2 are defined similarly. Prove that A_1A_2, B_1B_2 and C_1C_2 concur.

   First solution Apply a polar transform with center O. Now we have (we use new denotations) two triangles $A_1B_1C_1$ and $A_2B_2C_2$ such that the cevians A_1A_1', B_1B_1', and C_1C_1' in the first triangle and the cevians A_2A_2', B_2B_2', and C_2C_2' in the second triangle are all concurrent in O. Let P_a be the intersection of A_1A_2 and $A_1'A_2'$, define P_b and P_c similarly, now we wish to prove that P_a, P_b, and P_c are collinear.

   To this end, apply a projective transform that maps P_a and P_b to infinity. Then $OA_1' : A_1'A_1 = OA_2' : A_2'A_2$ and $OB_1' : B_1'B_1 = OB_2' : B_2'B_2$. However, $OA_1'/A_1'A_1 + OB_1'/B_1'B_1 + OC_1'/C_1'C_1 = S_{OB_1C_1}/S_{A_1B_1C_1} + S_{OC_1A_1}/S_{A_1B_1C_1} +$

$S_{OA_1B_1}/S_{A_1B_1C_1} = 1$ (signed areas) and similarly for the second triangle, so $OC_1' : C_1'C_1 = OC_2' : C_2'C_2$ and P_c is at infinity, too.

Second solution. Note that for any point X lying on C_1C_2 we have (the areas are directed)

$$S_{XAB}S_{A'B'C'} = S_{XA'B'}S_{ABC}.$$

To prove this equality it is sufficient to note that it is correct for C_1, C_2. Also it is easy to see that this equality is not true for all points of the plane, therefore this is the equation of line C_1C_2. Similarly we can find the equations of lines A_1A_2 and B_1B_2. It is evident that the point satisfying two of these three equations satisfy the third one.