XIV Geometrical Olympiad
in honour of I.F.Sharygin
Final round. Ratmino, 2018, July 31

Problems

First day. 8 grade

8.1. The incircle of right-angled triangle ABC ($\angle C = 90^\circ$) touches BC at point K. Prove that the chord of the incircle cut by the line AK is twice as large as the distance from C to that line.

8.2. A rectangle $ABCD$ and its circumcircle are given. Let E be an arbitrary point lying on the minor arc BC. The tangent to the circle at B meets CE at point G. The segments AE and BD meet at point K. Prove that GK and AD are perpendicular.

8.3. Let ABC be a triangle with $\angle A = 60^\circ$, and AA', BB', CC' be its internal angle bisectors. Prove that $\angle B'A'C' \leqslant 60^\circ$.

8.4. Find all sets of six points in the plane, no three collinear, such that if we partition such set arbitrarily into two sets of three points, then two obtained triangles are congruent.

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XIV Geometrical Olympiad

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Final round. Ratmino, 2018, August 1

Problems

Second day. 8 grade

8.5. The side AB of a square $ABCD$ is a base of an isosceles triangle ABE ($AE = BE$) lying outside the square. Let M be the midpoint of AE, O be the common point of AC and BD, and K be the common point of ED and OM. Prove that $EK = KO$.

8.6. The corresponding angles of quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ are equal. Also $AB = A_1B_1$, $AC = A_1C_1$, $BD = B_1D_1$. Are the quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ necessarily congruent?

8.7. Let ω_1, ω_2 be two circles centered at O_1, O_2 and lying each outside the other. Points C_1, C_2 lie on these circles in the same semiplane with respect to O_1O_2. The ray O_1C_1 meets ω_2 at points A_2, B_2, and the ray O_2C_2 meets ω_1 at points A_1, B_1. Prove that $\angle A_1O_1B_1 = \angle A_2B_2C_2$ if and only if $C_1C_2 \parallel O_1O_2$.

8.8. Let I be the incenter of fixed triangle ABC, and D be an arbitrary point on side BC. The perpendicular bisector of AD meets BI and CI at points F and E, respectively. Find the locus of orthocenters of triangles EIF.

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Problems

First day. 9 grade

9.1. Let M be the midpoint of AB in a right-angled triangle ABC with $\angle C = 90^\circ$. A circle passing through C and M intersects BC and AC at P and Q, respectively. Let c_1, c_2 be circles with centers P, Q and radii BP, AQ, respectively. Prove that c_1, c_2 and the circumcircle of ABC are concurrent.

9.2. A triangle ABC is given. A circle γ centered at A meets segments AB and AC. The common chord of γ and the circumcircle of ABC meets AB and AC at points X and Y respectively. The segments CX and BY meet γ at points S and T respectively. The circumcircles of triangles ACT and BAS meet at points A and P. Prove that CX, BY and AP concur.

9.3. The vertices of triangle DEF lie on different sides of triangle ABC. The lengths of the segments of the tangents from the incenter of DEF to the excircles of ABC are equal. Prove that $4S_{DEF} \geqslant S_{ABC}$. (By S_{XYZ} we denote the area of triangle XYZ.)

9.4. Let BC be a fixed chord of a given circle ω. Let A be a variable point on the major arc BC of ω. Let H be the orthocenter of triangle ABC. Points D and E lying on lines AB and AC respectively are such that H is the midpoint of segment DE . Let O_A be the circumcenter of triangle ADE. Prove that, as A varies, all points O_A lie on a fixed circle.

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Final round. Ratmino, 2018, August 1

Problems

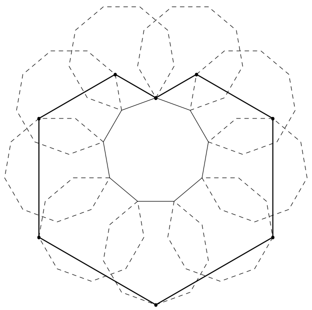
Second day. 9 grade

9.5. Let $ABCD$ be a cyclic quadrilateral, BL and CN be the internal angle bisectors in triangles ABD and ACD respectively. The circumcircles of triangles ABL and CDN meet at points P and Q. Prove that the line PQ passes through the midpoint of the arc AD not containing B.

9.6. Let $ABCD$ be a circumscribed quadrilateral. Prove that the common point of its diagonals, the incenter of triangle ABC and the center of excircle of triangle CDA touching the side AC are collinear.

9.7. Let B_1, C_1 be the midpoints of sides AC, AB of a triangle ABC, respectively. The tangents to the circumcircle at B and C meet the rays CC_1, BB_1 at points K and L respectively. Prove that $\angle BAK = \angle CAL$.

9.8. Consider a fixed regular n-gon of unit side. When a second regular n-gon of unit side rolls around the first one, one of its vertices successively pinpoints the vertices of a closed broken line κ as in the figure.



Let A be the area of a regular n-gon of unit side, and let B be the area of a regular n-gon of unit circumradius. Prove that the area enclosed by κ equals $6A - 2B$.

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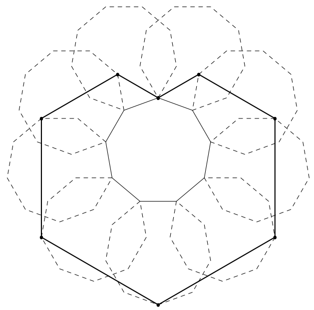

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Problems

First day. 10 grade

10.1. The altitudes AH, CH of an acute-angled triangle ABC meet the internal bisector of angle B at points L_1, P_1, and the external bisector of this angle at points L_2, P_2. Prove that the orthocenters of triangles HL_1P_1, HL_2P_2 and the vertex B are collinear.

10.2. A fixed circle ω is inscribed into an angle with vertex C. An arbitrary circle passes through C, touches ω externally and meets the sides of the angle at points A and B. Prove that the perimeters of all triangles ABC are equal.

10.3. A cyclic n-gon is given. The midpoints of all its sides are concyclic. The sides of the n-gon cut n arcs of this circle lying outside the n-gon. Prove that these arcs can be coloured red and blue in such a way that the sum of the lengths of red arcs is equal to the sum of the lengths of blue arcs.

10.4. We say that a finite set S of red and green points in the plane is *separable* if there exists a triangle δ such that all points of one colour lie strictly inside δ and all points of the other colour lie strictly outside of δ. Let A be a finite set of red and green points in the plane, in general position. Is it always true that if every 1000 points in A form a separable set then A is also separable?

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10.5. Let w be the incircle of a triangle ABC. The line passing through the incenter I and parallel to BC meets w at points A_B an A_C (A_B lies in the same semiplane with respect to AI as B). The lines BA_B and CA_C meet at point A_1. The points B_1 and C_1 are defined similarly. Prove that AA_1, BB_1 and CC_1 concur.

10.6. Let ω be the circumcircle of a triangle ABC, and KL be the diameter of ω passing through the midpoint M of AB (K and C lie on different sides of AB). A circle passing through L и M meets segment CK at points P and Q (Q lies on the segment KP). Let LQ meet the circumcircle of triangle KMQ again at point R. Prove that the quadrilateral $APBR$ is cyclic.

10.7. A convex quadrilateral $ABCD$ is circumscribed about a circle of radius r. What is the maximum possible value of $\frac{1}{AC^2}+\frac{1}{BD^2}$?

10.8. Two triangles ABC and $A'B'C'$ are given. The lines AB and $A'B'$ meet at point C_1, and the lines parallel to them and passing through C and C', respectively, meet at point C_2. The points A_1, A_2, B_1, B_2 are defined similarly. Prove that A_1A_2, B_1B_2 and C_1C_2 are either concurrent or parallel.

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