XIV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round. Solutions

1. (L.Shteingarts, grade 8) Three circles lie inside a square. Each of them touches externally two remaining circles. Also each circle touches two sides of the square. Prove that two of these circles are congruent.

   Solution. If two circles are inscribed into the same angle of the square, then the third one can not touch them and two sides. Hence we can suppose that the circles are inscribed into the angles A, B and C of square $ABCD$. But then two circles inscribed into angles A and C are symmetric with respect to diagonal BD, therefore they are congruent.

2. (N.Moskvitin, grade 8) A cyclic quadrilateral $ABCD$ is given. The lines AB and DC meet at point E, and the lines BC and AD meet at point F. Let I be the incenter of triangle AED, and a ray with origin F be perpendicular to the bisector of angle AID. In which ratio this ray dissects the angle AFB?

   Answer. $1:3$.

   Solution. Note that the angle between the bisectors of angles AED and AFB is equal to the semisum of angles FAE and FCE, i.e. $90°$. Thus the angle between the bisector of angle AFB and the ray FK, where K is the projection of F to the bisector of angle AID, is equal to $180° - \angle EIK = 180° - (90° + \angle A/2) - (180° - \angle A/2 - \angle D/2)/2 = (\angle D - \angle A)/4 = \angle AFB/4$ (fig.2), therefore $\angle AFK = \angle AFB/4$.

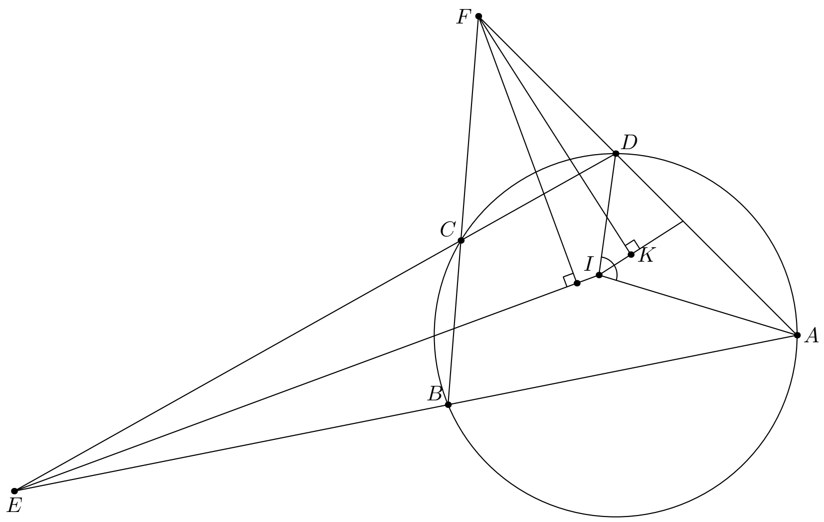


Fig. 2

3. (A.Zaslavsky, grade 8) Let AL be the bisector of triangle ABC, D be its midpoint, and E be the projection of D to AB. It is known that $AC = 3AE$. Prove that CEL is an isosceles triangle.

   Solution. Let F be the projection of L to AB, and G be the reflection of E about F. By the Thales theorem we have $AE = EF = FG$ and $AG = 3AE = AC$. Since AL is the bisector of angle A, and FL is the perpendicular bisector to segment EG, we obtain that $CL = LG = LE$ (fig.3).

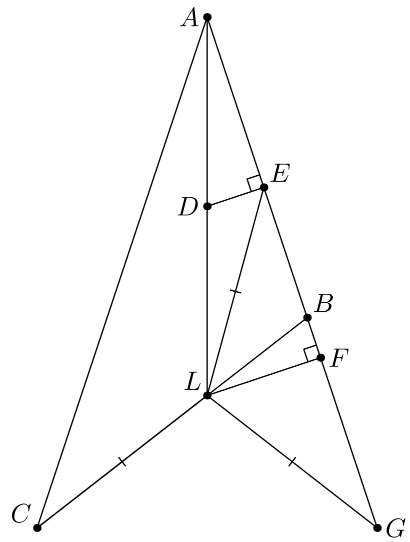


Fig. 3

4. (D.Shvetsov, grade 8) Let $ABCD$ be a cyclic quadrilateral. A point P moves along the arc AD which does not contain B and C. A fixed line l, perpendicular to BC, meets the rays BP, CP at points B_0, C_0 respectively. Prove that the tangent at P to the circumcircle of triangle PB_0C_0 passes through some fixed point.

   Solution. Let the tangent meet the circumcircle of $ABCD$ for the second time at point Q. Then $\angle BPQ = \angle B_0C_0P = 90^\circ - \angle BCP = 90^\circ - \angle BQP$ (fig.4). Thus $\angle PBQ = 90^\circ$, i.e. PQ is a diameter of circle $ABCD$. Hence all tangents pass through the center of this circle.

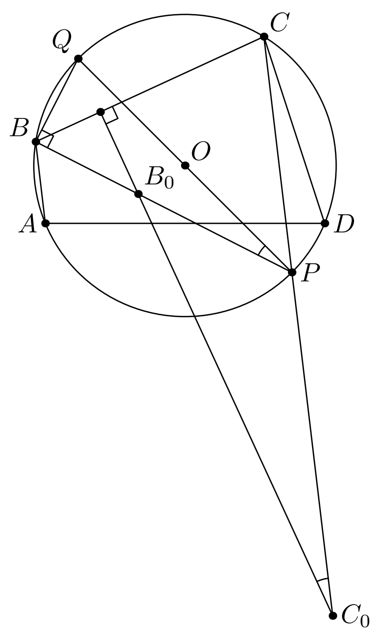


Fig. 4

5. (N.Moskvitin, grades 8–9) The vertex C of equilateral triangles ABC and CDE lies on the segment AE, and the vertices B and D lie on the same side with respect to this

segment. The circumcircles of these triangles centered at O_1 and O_2 meet for the second time at point F. The lines O_1O_2 and AD meet at point K. Prove that $AK = BF$.

Solution. Note that triangles ACD and BCE are congruent because $AC = BC$, $CD = CE$ and $\angle ACD = \angle BCE = 120°$. Also, since $\angle BFC = 120°$ and $\angle CFE = 60°$, we obtain that F lies on the segment BE. Finally triangles O_1CO_2 and ACD are similar, hence $\angle CO_1K = \angle CAK$, i.e the points A, O_1, K and C are concyclic (fig.5). Therefore $\angle ACK = 180° - \angle AO_1K = 60° - \angle CO_1K = 60° - \angle CBF = \angle BCF$, which yields the required equality.

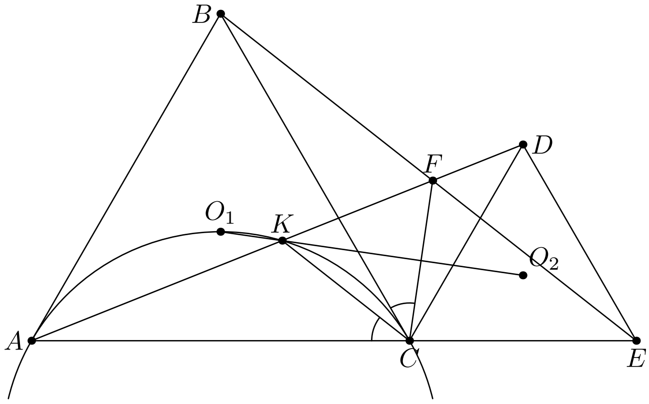


Fig. 5

6. (L.Shteingarts, grades 8–9) Let CH be the altitude of a right-angled triangle ABC ($\angle C = 90°$) with $BC = 2AC$. Let O_1, O_2 and O be the incenters of triangles ACH, BCH and ABC respectively, and H_1, H_2, H_0 be the projections of O_1, O_2, O respectively to AB. Prove that $H_1H = HH_0 = H_0H_2$.

   Solution. Similarity of triangles HAC and HCB implies $HO_2 = 2HO_1$, thus $HH_2 = HH_1$. So we have to prove that $H_1H_0 = 2H_0H_2$. But $H_1H_0 = AH_0 - AH_1 = AH_0(AB - AC)/AB = (AB + AC - BC)(AB - AC)/2AB = (BC^2 - BC(AB - AC))/2AB = BC(AC + BC - AB)/2AB$. Similarly $H_0H_2 = AC(AC + BC - AB)/2AB$, and we obtain the required equality.

7. (I.Spiridonov, grades 8–9) Let E be a common point of circles w_1 and w_2. Let AB be a common tangent to these circles, and CD be a line parallel to AB, such that A and C lie on w_1, B and D lie on w_2. The circles ABE and CDE meet for the second time at point F. Prove that F bisects one of arcs CD of circle CDE.

   Solution. Let the lines AC and BF meet at point H, and the lines BD and AF meet at point G.

   Since AB touches the circumcircle of triangle CAE, we have $(CA, CE) = (AB, AE)$. Since $ABEF$ is a cyclic quadrilateral, we have $(AB, AE) = (FB, FE)$. Then

$$(CH, CE) = (CA, CE) = (AB, AE) = (FB, FE) = (FH, FE)$$

$$(CH, CE) = (FH, FE)$$

   and $CHFE$ is a cyclic quadrilateral. Similarly we obtain that $DGFE$ is cyclic. Since $CFED$ is cyclic, we obtain that C, D, E, F, H, G are concyclic (fig.7).

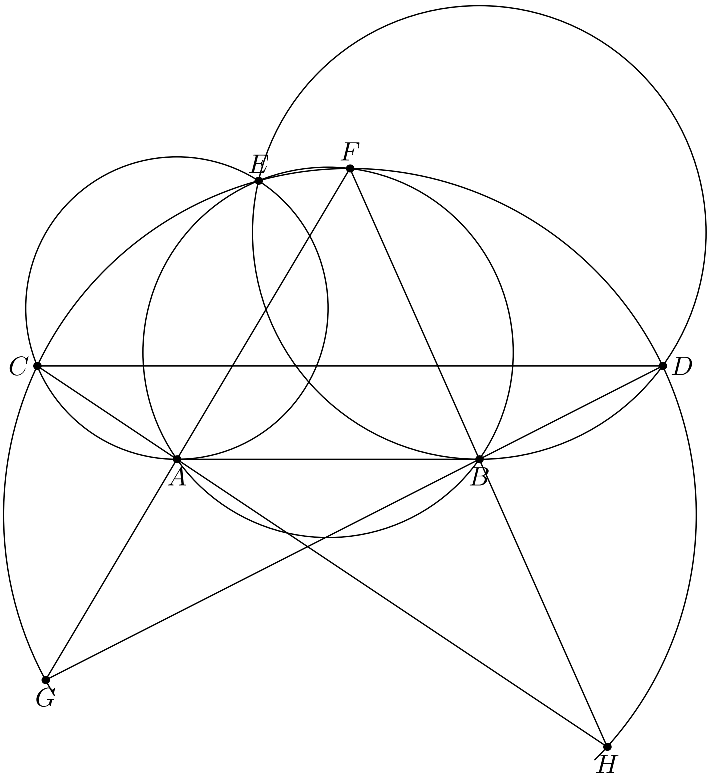


Fig. 7

Applying the Pascal theorem to cyclic hexagon $FFHCDG$ we obtain that the common points of FF and CD, FH and DG, HC and GF are collinear (line FF is the tangent at F to the circle CFD, denote this line by l). So A, B and the common point of l and CD are collinear. But $AB \parallel CD$, therefore $l \parallel CD$ and F is the midpoint of arc CD.

8. (K.Kadyrov, grades 8–9) Restore a triangle ABC by the Nagel point, the vertex B and the foot of the altitude from this vertex.

   Solution. Since the centroid of the triangle divides the segment between the Nagel point N and the incenter as $2:1$, we can find the radius of the incircle (we know the altitude and the distance from N to the base). Now, using the formulas for the area $S = bh_b/2 = pr = (p-b)r_b$, we can find the radius r_b of the excircle. Since the excircle touches the base at its common point with BN, we can construct this circle, draw the tangents to it from B, and restore the triangle.

9. (B.Frenkin, grades 8–9) A square is inscribed into an acute-angled triangle: two vertices of this square lie on the same side of the triangle and two remaining vertices lies on two remaining sides. Two similar squares are constructed for the remaining sides. Prove that three segments congruent to the sides of these squares can be the sides of an acute-angled triangle.

   Solution. Consider the greatest of three squares. Let its vertices K, L lie on AB, and the vertices M, N lie on BC, AC respectively. Draw the perpendiculars MX, NY to AC, BC respectively and the line passing through M, parallel to AC and meeting AB at point Z (fig.9). Since $MX < MN = ML < MZ$, the side of the inscribed square having the base on AC is greater than MX. Similarly the side of the square having the base on BC is greater than NY. Since $MN^2 - MX^2 = NX^2 < NY^2$, the triangle with the sidelengths MN, MX, NY is acute-angled. The more so, the sides of the three squares can form an acute-angled triangle.

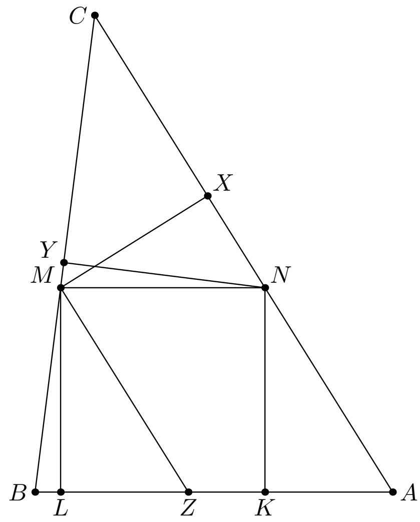


Fig. 9

10. (Folklore, grades 8–9) In the plane, 2018 points are given such that all distances between them are different. For each point, mark the closest one of the remaining points. What is the minimal number of marked points?

    Answer. 449.

    Solution. Divide the points into classes such that all points of the same class have the same closest point. Note that each class contains at most five points. In fact, if B is closest for $A_1, A_2, \dots, A_n$ then A_1A_2 is the greatest side of triangle A_1A_2B, thus $\angle A_1BA_2 > 60°$. The same is true for the angles $A_2BA_3, \dots, A_nBA_1$. Thus $n \leq 5$. A similar argument shows that if B is closest to $A_1, \dots, A_5$ then one of these points is closest to B and the class of B contains less than five points. Therefore at least $2n/9$ points of n points have to be marked, and for $n = 2018$ we have at least 449 marked points.

    On the other hand, consider the configuration of 9 points on fig.10. Point A is closest for five points marked by a circ, and point B is closest for four points marked by a square. Take now 224 such groups, placed at a great distance one from another, and to one of them add two points such that C is closest to them. In this configuration $223 \cdot 2 + 3 = 449$ points are marked.

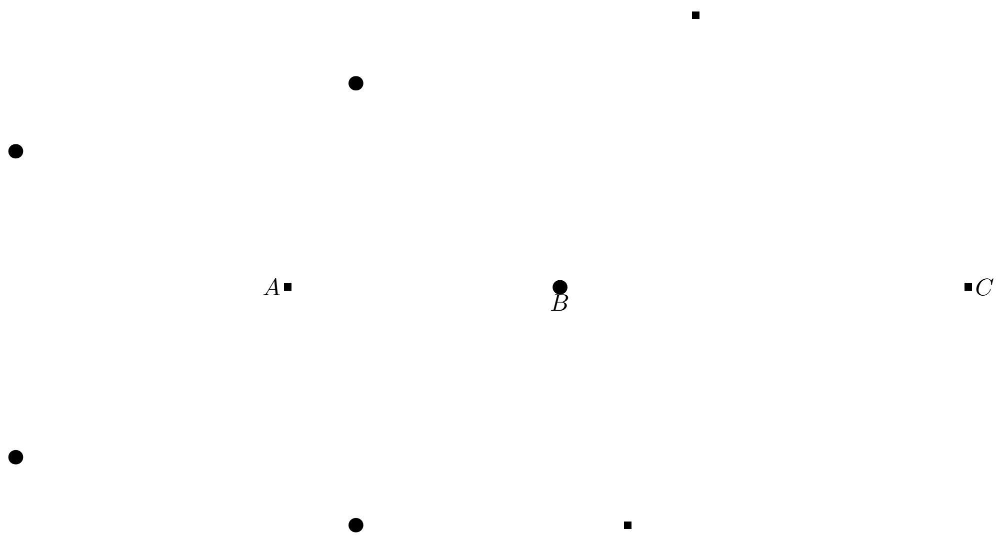


Fig. 10

11. (A.Zaslavsky, grades 8–9) Let I be the incenter of a nonisosceles triangle ABC. Prove that there exists a unique pair of points M, N lying on the sides AC, BC respectively, such that $\angle AIM = \angle BIN$ and $MN \parallel AB$.

Solution. Consider the lines passing through A and B and parallel to IM, IN respectively. Since $MN \parallel AB$, their common point J lies on the ray CI and $\angle IAJ = \angle IBJ$. Thus the radii of circles AIJ and BIJ are equal, i.e. these circles are symmetric with respect to IJ. Hence the circle AIJ passes through the reflection B' of B about the bisector of angle C (fig.11). But A, B, I and B' are concyclic. Therefore the circles AIJ and BIJ coincide and J is the excenter of the triangle. Then $\angle AIM = \angle BIN = 90°$.

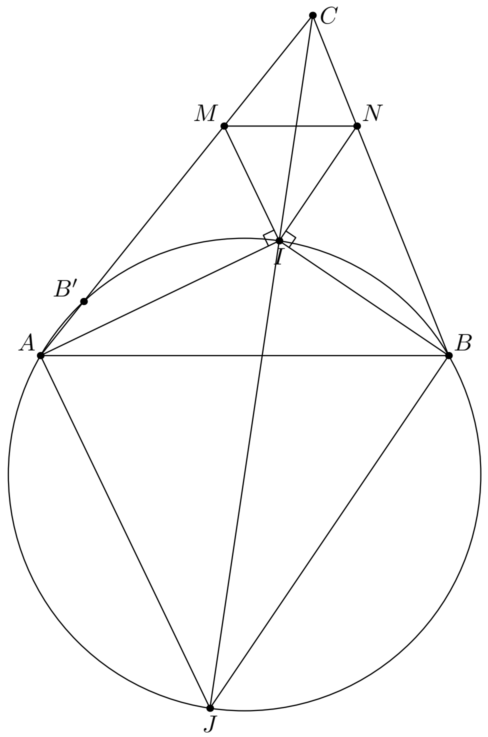


Fig. 11

12. (A.Didin, grades 8–9) Let BD be the external bisector of a triangle ABC with $AB > BC$; K and K_1 be the touching points of side AC with the incicrle and the excircle centered at I and I_1 respectively. The lines BK and DI_1 meet at point X, and the lines BK_1 and DI meet at point Y. Prove that $XY \perp AC$.

Solution. Since I and I_1 lie on the bisector of angle B, we have $BD \perp BI$. Hence B, K lie on the circle with diameter BI, and B, K_1 lie on the circle with diameter BI_1. Therefore $\angle YDK = \angle IBX$, $\angle YBI_1 = \angle KDX$, $\angle YBX = \angle YDX$ and points B, D, X, Y are concyclic (fig.12). Thus $\angle XYD = \angle XBD = 90^\circ - \angle YDK$, i.e. $XY \perp AC$.

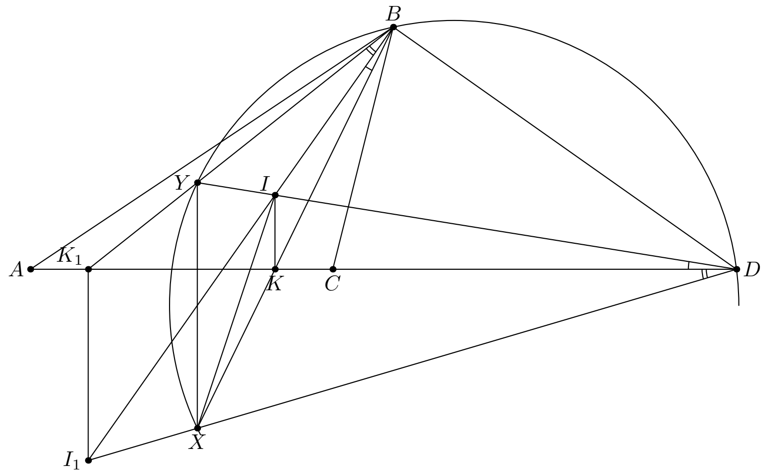


Fig. 12

13. (G.Feldman, grades 9–11) Let $ABCD$ be a cyclic quadrilateral, and M, N be the midpoints of arcs AB and CD respectively. Prove that MN bisects the segment between the incenters of triangles ABC and ADC.

Solution. Clearly the incenters I, J of triangles ABC and ADC lie on the segments CM and AN respectively. Also by the trident theorem $IM = AM = 2R\sin\angle ANM$. Thus the distance from I to MN is equal to $IM\sin\angle NMC = 2R\sin\angle ANM\sin\angle NMC$. We obtain the same expression for the distance from J to MN. Since I and J lie on the opposite sides from MN, this equality yields the assertion of the problem (fig.13).

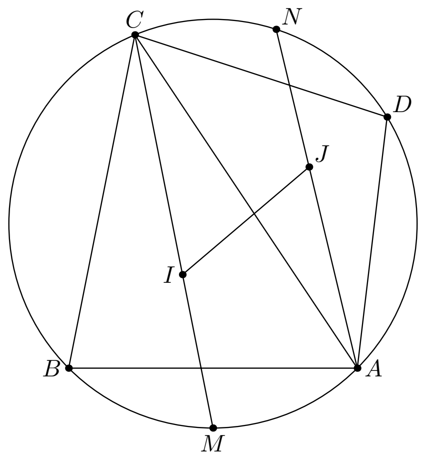


Fig. 13

14. (M.Kungozhin, grades 9–11) Let ABC be a right-angled triangle with $\angle C = 90°$, K, L, M be the midpoints of sides AB, BC, CA respectively, and N be a point of side AB. The line CN meets KM and KL at points P and Q respectively. Points S, T lying on AC and BC respectively are such that $APQS$ and $BPQT$ are cyclic quadrilaterals. Prove that

a) if CN is a bisector, then CN, ML and ST concur;

b) if CN is an altitude, then ST bisects ML.

Solution. a) By the assumption we have $CP = CM\sqrt{2} = AC/\sqrt{2}$, $CQ = BC/\sqrt{2}$. Hence $CS = CP \cdot CQ/AC = BC/2 = BL$. Similarly $CT = CM$. Therefore the segments ML and ST are symmetric with respect to CN and meet on this line.

b) From the similarity of triangles CMP, QLC and ACB we obtain that $CP = AC \cdot AB/2BC$, $CQ = BC \cdot AB/2AC$. Thus $CS = AB^2/4AC$, $CT = AB^2/4BC$ and the triangle CST is similar to CBA. Therefore ST is perpendicular to the median of ABC, and since the altitude of triangle CST is equal to $AB/4$, we obtain that its foot coincides with the midpoint of ML.

15. (D.Hilko, grades 9–11) The altitudes AH_1, BH_2, CH_3 of an acute-angled triangle ABC meet at point H. Points P and Q are the reflections of H_2 and H_3 with respect to H. The circumcircle of triangle PH_1Q meets for the second time BH_2 and CH_3 at points R and S. Prove that RS is a medial line of triangle ABC.

Solution. Consider the common point R' of the medial line M_2M_3 and the altitude BH_2. Prove that R' lies on the circumcircle of triangle PH_1Q.

Since BH_3H_2C is a cyclic quadrilateral, we have $H_3H \cdot HC = H_2H \cdot HB$. Then $HB \cdot HP = HC \cdot HQ$ and $PBQC$ is a cyclic quadrilateral. Therefore $\angle H_2PQ = \angle BCQ = \angle BAH_1$. Also, since R' lies on the medial line of ABC, we obtain that $\angle H_1AR' = \angle AH_1R'$. Now the triangles H_3HH_1 and BM_3R' are similar because $\angle M_3BR' = \angle H_3H_1H$, and $\angle M_3R'B = \angle HH_3H_1$. Thus

$$\frac{H_3H_1}{M_3R'} = \frac{HH_1}{BM_3}.$$

Since $H_3H = HQ$ and $BM_3 = M_3A$, we have

$$\frac{QH}{M_3R'} = \frac{HH_1}{AM_3}.$$

Clearly $\angle QHH_1 = \angle B = \angle AM_3R'$. This implies that triangles AM_3R' and HH_1Q are similar, hence $\angle HH_1Q = \angle M_3AR'$. Then $\angle QH_1R' = \angle HH_1Q - \angle HH_1R' = \angle M_3AR' - \angle R'AH_1 = \angle BAH_1 = \angle R'PQ$. Therefore PH_1QR' is a cyclic quadrilateral (fig.15). Hence $R = R'$, i.e R lies on the medial line of ABC. Similarly S lies on the medial line and we obtain the required assertion.

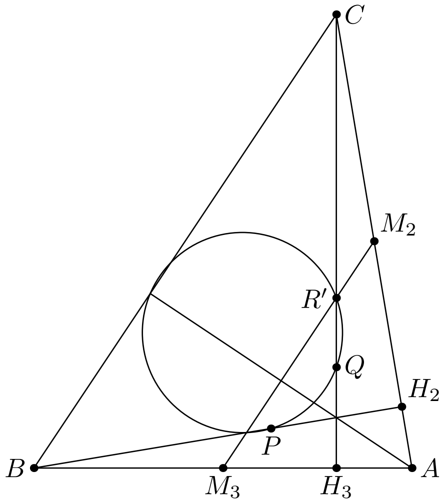


Fig. 15

16. (P.Ryabov, grades 9–11) Let ABC be a triangle with $AB < BC$. The bisector of angle C meets the line parallel to AC and passing through B, at point P. The tangent at B to the circumcircle of ABC meets this bisector at point R. Let R' be the reflection of R with respect to AB. Prove that $\angle R'PB = \angle RPA$.

Solution. Since the lines BR and BP are symmetric with respect to the bisector of angle B, we obtain that P and R are isogonally conjugated with respect to ABC. Thus $\angle R'AB = \angle RAB = \pi - \angle CAP$, i.e. lines AR' and AC are symmetric with respect to the bisector of angle A. Similarly BR' and BC are symmetric with respect to the bisector of angle B. Therefore R' and C are isogonally conjugated with respect to triangle ABP, which yields the required assertion (fig.16).

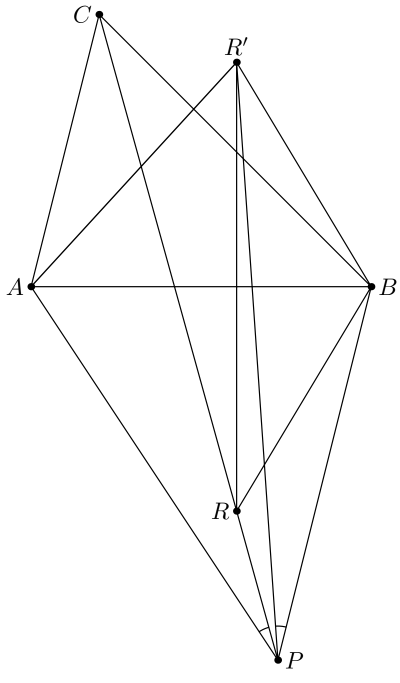


Fig. 16

17. (S.Takhaev, grades 10–11) Let each of circles α, β, γ touches two remaining circles externally, and all of them touch a circle Ω internally at points A_1, B_1, C_1 respectively. The common internal tangent to α and β meets the arc A_1B_1 not containing C_1 at point C_2. Points A_2, B_2 are defined similarly. Prove that the lines A_1A_2, B_1B_2, C_1C_2 concur.

   Solution. Let the tangents to Ω at A_1, B_1, C_1 form a triangle ABC. Without loss of generality suppose that Ω is the incircle (not the excircle) of this triangle. Note that, for example, C is the radical center of circles α, β and Ω, i.e. C lies on the common internal tangent to α and β. Also the common tangents to α, β, γ concur at their radical center, denote it by X. Hence we can reformulate the assertion of the problem as follows.

   A triangle ABC and a point X inside its incircle are given. The segments XA, XB, XC meet the incircle at A_2, B_2, C_2 respectively, and the sides BC, CA, AB touch it at A_1, B_1, C_1. Then A_1A_2, B_1B_2 and C_1C_2 concur.

   Applying the sinus theorem to triangles A_1CC_2 and B_1CC_2 we obtain

$$\frac{A_1C_2}{B_1C_2} = \frac{\sin\angle A_1CC_2}{\sin\angle B_1CC_2}\cdot\frac{\sin\angle CB_1C_2}{\sin\angle CA_1C_2} = \frac{\sin\angle A_1CC_2}{\sin\angle B_1CC_2}\cdot\frac{B_1C_2}{A_1C_2}.$$

   Now applying the Ceva theorem to triangles ABC and $A_1B_1C_1$ we obtain the required assertion.

18. (A.Polyanskii, N.Polyanskii, grades 10–11) Let C_1, A_1, B_1 be points on sides AB, BC, CA of triangle ABC, such that AA_1, BB_1, CC_1 concur. The rays B_1A_1 and B_1C_1 meet the circumcircle of the triangle at points A_2 and C_2 respectively. Prove that A, C, the common point of A_2C_2 and BB_1 and the midpoint of A_2C_2 are concyclic.

   Solution. Let K be the common point of A_2C_2 and AC, M be the midpoint of A_2C_2, and N be the second common point of circle ACM with A_2C_2. Then $KM\cdot KN = KA\cdot KC = KA_2\cdot KC_2$, i.e the quadruple A_2, C_2, K, N is harmonic. Projecting A_2C_2 from B_1 to AA_1 we obtain that A_1, the common point of AA_1 with B_1C_1, A and the common point of BN with AA_1 also form a harmonic quadruple. Thus BN passes through the common point of AA_1, BB_1 and CC_1, i.e. coincides with BB_1.

19. (A.Myakishev, grades 10–11) Let a triangle ABC be given. On a ruler three segment congruent to the sides of this triangle are marked. Using this ruler construct the orthocenter of the triangle formed by the tangency points of the sides of ABC with its incircle.

   Solution. On the extension of AC beyond C, construct the segment $CX = BC$. We obtain the line BX parallel to the bisector of angle C. Similarly we can construct the line parallel to the bisector of C and passing through A, and having two parallel lines we can draw a line parallel to them and passing through an arbitrary point. Hence it is sufficient to construct the touching points A', B', C' of the incircle with BC, CA, AB (the bisectors of ABC are perpendicular to the sides of $A'B'C'$). On the extensions of AB beyond A and B, construct the segments $AU = BC$ and $BV = AC$ respectively. Since $AC' = s - BC$, where s is the semiperimeter of ABC, we obtain that C' is the midpoint of UV. Drawing the lines through U and V parallel to two bisectors, we obtain a parallelogram with diagonal UV, and drawing its second diagonal we find C'.

20. (A.Zimin, grades 10–11) Let the incircle of a nonisosceles triangle ABC touch AB, AC and BC at points D, E and F respectively. The corresponding excircle touches the side

BC at point N. Let T be the common point of AN and the incircle, closest to N, and K be the common point of DE and FT. Prove that $AK||BC$.

Solution. Let G be the point of the incircle opposite to F. Since the incircle and the excircle are homothetic with center A, we obtain that A, G and N are collinear, and $FT \perp AN$. The polar transformation with respect to the incircle maps ED into A, maps FT into the common point of tangents at F and T, i.e. the common midpoint of FN and BC, and maps the line through A parallel to BC into the common point L of ED and GF. Thus we have to prove that AL is a median of ABC (fig.20).

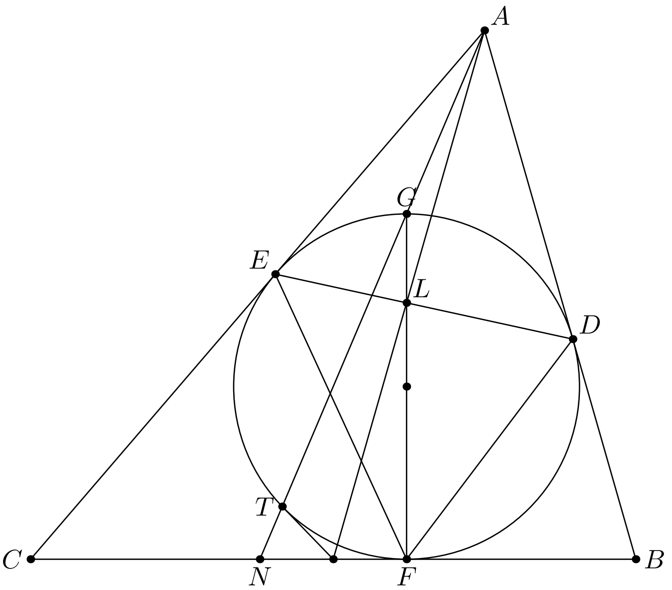


Fig. 20

Since $AE = AD$, we have $\sin\angle CAL : \sin\angle BAL = EL : DL$. Applying the sinus theorem to triangles EFL and EDL, we obtain $EL : DL = EF \sin\angle EFL : DF \sin\angle DFL$. But $\angle EFL = \angle C/2$, $\angle DFL = \angle B/2$, and $EF : DF = \cos\angle C/2 : \cos\angle B/2$. Therefore $\sin\angle CAL : \sin\angle BAL = AB : AC$, i.e. AL is a median.

21. (proposed by B.Frenkin, grades 10–11) In the plane a line l and a point A outside it are given. Find the locus of the incenters of acute-angled triangles having a vertex A and an opposite side lying on l.

Solution. Let H be the projection of A to l. Since a triangle is acute-angled, we obtain that its incenter I and one of its vertices, for example B, lie on the opposite sides with respect to AH. Hence the distance from I to AH is less than the distance from I to AB which is equal to the inradius r, i.e. the distance from I to l. Therefore I lies inside the right angle formed by the bisectors of two angles between AH and l. Also it is clear that $r < AH/2$, i.e. I lies inside a strip bounded by l and the perpendicular bisector to AH. Finally, since angle A is acute, we have $AI = r/\sin\angle A/2 > r\sqrt{2}$, hence I lies between the branches of an equilateral hyperbola with focus A and directrix l. On the other hand, for an arbitrary point satisfying these conditions we can construct the circle centered at this point and touching l, draw the tangents to it from A and obtain an acute-angled triangle. So the required locus is bounded by the bisectors of angles between l and AH, the perpendicular bisector to AH and the corresponding branch of the hyperbola (the bounds are not included).

22. (N.Beluhov, grades 10–11) Six circles of unit radius lie in the plane so that the distance between the centers of any two of them is greater than d. What is the least value of d such that there always exists a straight line which does not intersect any of the circles and separates the circles into two groups of three?

    Solution. Let $O_1O_2O_3$ be an equilateral triangle of side d, O_4 be such that $O_1O_4 = d$ and $\angle O_2O_1O_4 = \angle O_4O_1O_3 = 150°$, and O_5 and O_6 be defined analogously so that the complete figure is rotationally symmetric about the center of $\triangle O_1O_2O_3$. The six circles centered at O_1, O_2, $\ldots$, O_6 show that $d \geq \frac{2}{\sin 15°} = 2(\sqrt{2} + \sqrt{6})$.

    Put $d = \frac{2}{\sin 15°}$. Let us show that a halving line always exists.

    Enumerate the circles' centers from 1 to 6, and let l be a straight line such that the six centers' projections onto l are distinct. The order of the projections from left to right gives us a permutation σ of the numbers 1 through 6.

    Rotate l counterclockwise until it makes a complete 360° turn. Each time that l becomes perpendicular to a line through two centers (it suffices to consider the case when no three centers are collinear), two neighbouring elements of σ switch their positions. Since there are $\binom{6}{2} = 15$ such lines, $2 \cdot 15 = 30$ such transpositions occur.

    We say that a transposition is *external* if, at the time when it takes place, the two centers involved are either the first two or the last two elements of σ (i.e., $AB \circ\circ\circ\circ \to BA \circ\circ\circ\circ$ or $\circ\circ\circ\circ AB \to \circ\circ\circ\circ BA$). Otherwise, we say that a transposition is *internal*.

    Since an external transposition corresponds to a side of the centers' convex hull, there are at least $2 \cdot 3 = 6$ external transpositions and at most $30 - 6 = 24$ internal ones.

    Since $\frac{360°}{24} = 15°$, there is some interval s of the rotation of l having length at least 15°, containing no internal transpositions. This means that throughout s both the third and the fourth element of σ remain fixed. Let A and B be those elements.

    Consider the strip L bounded by the lines through A and B perpendicular to l. Throughout s, L does not contain any centers apart from A and B. Since the length of s is at least 15°, there is some position of L during s such that the acute angle between AB and L is at least 15° and, consequently, the width of L is greater than two. The midline of this instance of L does the job.

23. (A.Kanel-Belov, 10–11) (grades 10–11) The plane is divided into convex heptagons with diameters less than 1. Prove that an arbitrary disc with radius 200 intersects most than a billion of them.

    Solution. Consider a disc K with radius R. Let k vertices of heptagons lie inside K. The average angle at these vertices is at most $2\pi/3$. (If a vertex is common for more than three heptagons then the average angle is less than $2\pi/3$, and if a vertex lies on a side then the average angle is at most $\pi/2$).

    On the other hand, consider the heptagons lying inside K or intersecting the bounding circle of K. Their average angle is $5\pi/7$. Let n of their vertices lie outside K, all of them lie at a distance not greater than 1 from K. Each angle in such vertex is less than π (of course this is true for any angle of a convex polygon).

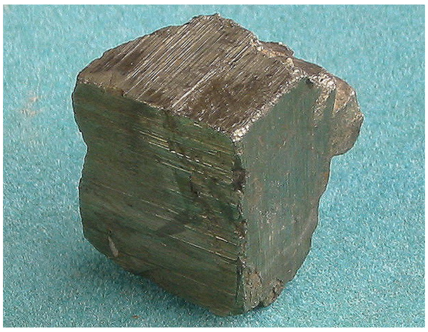
To satisfy the balance, the inequality $n\pi + k \cdot 2\pi/3 > (n + k)5\pi/7$ is needed. Hence $n > k/6$.

So the number of vertices lying at a distance not greater than 1 from K is greater than the number of vertices inside K divided by 6.

Now note that $(1 + 1/6)^6 > 2$, $(1 + 1/6)^{60} > 2^{10} > 1000$ and $(1 + 1/6)^{180} > 1000^3$.

Hence the number of heptagons (the number of angles divided by 7) intersecting a disc with radius 200 is at least 10^9.

24. (A.Solynin, grades 10–11) A crystal of pyrite is a parallelepiped with dashed faces.



The dashes on any two adjacent faces are perpendicular. Does there exist a convex polytope with the number of faces not equal to 6, such that its faces can be dashed in such a manner?

Answer. Yes.

Solution. Take a quadrilateral $ABCD$. Let the lines AB and CD meet at point X, and the lines AD and BC meet at point Y. Draw the plane passing through the line XY and perpendicular to the plane $ABCD$, and a point S on this plane such that $\angle XSY = 90°$. Now dash the faces SAB and SCD of pyramid $SABCD$ by lines parallel to SX, dash the faces SBC and SCD by lines parallel to SY, and dash the face $ABCD$ by perpendiculars to the plane SXY. Clearly the obtained dashes satisfy the condition.