XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. First day. 8 form
Ratmino, July 30, 2019

1. A trapezoid with bases AB and CD is inscribed into a circle centered at O. Let AP and AQ be the tangents from A to the circumcircle of triangle CDO. Prove that the circumcircle of triangle APQ passes through the midpoint of AB.

2. A point M inside triangle ABC is such that $AM = AB/2$ and $CM = BC/2$. Points C_0 and A_0 lying on AB and CB respectively are such that $BC_0 : AC_0 = BA_0 : CA_0 = 3$. Prove that the distances from M to C_0 and to A_0 are equal.

3. Construct a regular triangle using a plywood square. (*You can draw lines through pairs of points lying on the distance not greater than the side of the square, construct the perpendicular from a point to a line if the distance between them does not exceed the side of the square, and measure segments on the constructed lines equal to the side or to the diagonal of the square.*)

4. Let O and H be the circumcenter and the orthocenter of an acute-angled triangle ABC with $AB < AC$. Let K be the midpoint of AH. The line through K perpendicular to OK meets AB and the tangent to the circumcircle at A at points X and Y respectively. Prove that $\angle XOY = \angle AOB$.

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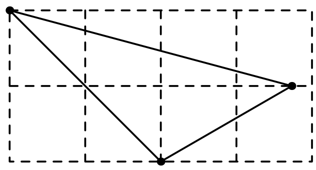
Final round. Second day. 8 form

Ratmino, July 31, 2019

5. A triangle having one angle equal to 45° is drawn on the chequered paper (see.fig.). Find the values of its remaining angles.



6. A point H lies on the side AB of regular pentagon $ABCDE$. A circle with center H and radius HE meets the segments DE and CD at points G and F respectively. It is known that $DG = AH$. Prove that $CF = AH$.

7. Let points M and N lie on the sides AB and BC of triangle ABC in such a way that $MN \parallel AC$. Points M' and N' are the reflections of M and N about BC and AB respectively. Let $M'A$ meet BC at X, and $N'C$ meet AB at Y. Prove that A, C, X, Y are concyclic.

8. What is the least positive integer k such that, in every convex 1001-gon, the sum of any k diagonals is greater than or equal to the sum of the remaining diagonals?

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. First day. 9 form
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1. A triangle OAB with $\angle A = 90^\circ$ lies inside a right angle with vertex O. The altitude of OAB from A is extended beyond A until it intersects the side of angle O at M. The distances from M and B to the second side of angle O are equal to 2 and 1 respectively. Find the length of OA.

2. Let P lie on the circumcircle of triangle ABC. Let A_1 be the reflection of the orthocenter of triangle PBC about the perpendicular bisector to BC. Points B_1 and C_1 are defined similarly. Prove that A_1, B_1, and C_1 are collinear.

3. Let $ABCD$ be a cyclic quadrilateral such that $AD = BD = AC$. A point P moves along the circumcircle ω of $ABCD$. The lines AP and DP meet the lines CD and AB at points E and F respectively. The lines BE and CF meet at point Q. Find the locus of Q.

4. A ship tries to land in the fog. The crew does not know the direction to the land. They see a lighthouse on a little island, and they understand that the distance to the lighthouse does not exceed 10 km (the precise distance is not known). The distance from the lighthouse to the land equals 10 km. The lighthouse is surrounded by reefs, hence the ship cannot approach it. Can the ship land having sailed the distance not greater than 75 km? (The waterside is a straight line, the trajectory has to be given before the beginning of the motion, after that the autopilot navigates the ship according to it.)

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5. Let R be the circumradius of a cyclic quadrilateral $ABCD$. Let h_1 and h_2 be the altitudes from A to BC and CD respectively. Similarly h_3 and h_4 are the altitudes from C to AB and AD. Prove that
$$\frac{h_1 + h_2 - 2R}{h_1 h_2} = \frac{h_3 + h_4 - 2R}{h_3 h_4}.$$

6. A non-convex polygon has the property that every three consecutive its vertices form a right-angled triangle. Is it true that this polygon has always an angle equal to 90° or to 270°?

7. Let the incircle ω of triangle ABC touch AC and AB at points E and F respectively. Points X, Y of ω are such that $\angle BXC = \angle BYC = 90^\circ$. Prove that EF and XY meet on the medial line of ABC.

8. A hexagon $A_1A_2A_3A_4A_5A_6$ has no four concyclic vertices, and its diagonals A_1A_4, A_2A_5 and A_3A_6 concur. Let l_i be the radical axis of circles $A_iA_{i+1}A_{i-2}$ and $A_iA_{i-1}A_{i+2}$ (the points A_i and A_{i+6} coincide). Prove that l_i, $i = 1, \ldots, 6$, concur.

XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. First day. 10 form
Ratmino, July 30, 2019

1. Given a triangle ABC with $\angle A = 45^\circ$. Let A' be the antipode of A in the circumcircle of ABC. Points E and F on segments AB and AC respectively are such that $A'B = BE$, $A'C = CF$. Let K be the second intersection of circumcircles of triangles AEF and ABC. Prove that EF bisects $A'K$.

2. Let A_1, B_1, C_1 be the midpoints of sides BC, AC and AB of triangle ABC, AK be its altitude from A, and L be the tangency point of the incircle γ with BC. Let the circumcircles of triangles LKB_1 and A_1LC_1 meet B_1C_1 for the second time at points X and Y respectively and γ meet this line at points Z and T. Prove that $XZ = YT$.

3. Let P and Q be isogonal conjugates inside triangle ABC. Let ω be the circumcircle of ABC. Let A_1 be a point on arc BC of ω satisfying $\angle BA_1P = \angle CA_1Q$. Points B_1 and C_1 are defined similarly. Prove that AA_1, BB_1, and CC_1 are concurrent.

4. Prove that the sum of two nagelians is greater than the semiperimeter of the triangle. (A nagelian is the segment between a vertex of a triangle and the tangency point of the opposite side with the corresponding excircle.)

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5. Let AA_1, BB_1, CC_1 be the altitudes of triangle ABC; and A_0, C_0 be the common points of the circumcircle of triangle A_1BC_1 with the lines A_1B_1 and C_1B_1 respectively. Prove that AA_0 and CC_0 meet on the median of ABC or are parallel to it.

6. Let AK and AT be the bisector and the median of an acute-angled triangle ABC with $AC > AB$. The line AT meets the circumcircle of ABC at point D. Point F is the reflection of K about T. If the angles of ABC are known, find the value of angle FDA.

7. Let P be an arbitrary point on side BC of triangle ABC. Let K be the incenter of triangle PAB. Let the incircle of triangle PAC touch BC at F. Point G on CK is such that $FG \parallel PK$. Find the locus of G.

8. Several points and planes are given in the space. It is known that for any two of given points there exist exactly two planes containing them, and each given plane contains at least four of given points. Is it true that all given points are collinear?