XV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round

Below is the list of problems for the first (correspondence) round of the XV Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four eldest grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem costs 7 points. A partial solution costs from 1 to 6 points. A text without significant advancement costs 0 points. The result of a participant is the sum of all obtained marks.

First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. **If a problem has an explicit answer, this answer must be presented distinctly**. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The **solutions** for the problems (in Russian or in English) must be **delivered not before December 1, 2018 and not later than on March 1, 2019**. To upload your work, enter the site **https://contest.yandex.ru/geomshar/**, indicate the language (English) in the right upper part of the page, press "Registration"in the left upper part, and follow the instructions.

Attention:

1. The solution of each problem (and of each part of it if any) must be contained in a **separate** pdf, doc, docx or jpg file. If the solution is contained in several files then pack them to an archive (zip or rar) and load it.

2. We recommend to prepare the paper using computer or to scan it rather than to photograph it. **In all cases, please check readability of the file before uploading**.

3. If you upload the solution of some problem more than once then only the last version is retained in the checking system. **Thus if you need to change something in your solution then you have to upload the whole solution again.**

If you have any technical problems with uploading of the work, apply to **geomshar@yandex.ru (DON'T SEND your work to this address)**.

The final round will be held in July–August 2019 in Moscow region. The winners of the correspondence round are invited to it if they don't graduate from school before. (For instance, if the last grade is 12 then we invite winners from 9–11 grades, and from 12 grade if they finish their school education later.) The graduates, winners of the correspondence round, will be awarded by diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** at the end of May 2019 at latest. If you want to know your detailed results, please use e-mail **geomshar@yandex.ru**.

1. (8) Let AA_1, CC_1 be the altitudes of triangle ABC, and P be an arbitrary point of side BC. Point Q on the line AB is such that $QP = PC_1$, and point R on the line AC is such that $RP = CP$. Prove that QA_1RA is a cyclic quadrilateral.

2. (8) The circle ω_1 passes through the center O of the circle ω_2 and meets it at points A and B. The circle ω_3 centered at A with radius AB meets ω_1 and ω_2 at points C and D (distinct from B). Prove that C, O, D are collinear.

3. (8) The rectangle $ABCD$ lies inside a circle. The rays BA and DA meet this circle at points A_1 and A_2. Let A_0 be the midpoint of A_1A_2. Points B_0, C_0, D_0 are defined similarly. Prove that $A_0C_0 = B_0D_0$.

4. (8) The side AB of triangle ABC touches the corresponding excircle at point T. Let J be the center of the excircle inscribed into angle A, and M be the midpoint of AJ. Prove that $MT = MC$.

5. (8–9) Let A, B, C and D be four points in general position, and ω be a circle passing through B and C. A point P moves along ω. Let Q be the common point of circles ABP and PCD distinct from P. Find the locus of points Q.

6. (8–9) Two quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ are mutually symmetric with respect to the point P. It is known that A_1BCD, AB_1CD and ABC_1D are cyclic quadrilaterals. Prove that the quadrilateral $ABCD_1$ is also cyclic.

7. (8–9) Let AH_A, BH_B, CH_C be the altitudes of the acute-angled triangle ABC. Let X be an arbitrary point of segment CH_C, and P be the common point of circles with diameters H_CX and BC, distinct from H_C. The lines CP and AH_A meet at point Q, and the lines XP and AB meet at point R. Prove that A, P, Q, R, H_B are concyclic.

8. (8–9) The circle ω_1 passes through the vertex A of the parallelogram $ABCD$ and touches the rays CB, CD. The circle ω_2 touches the rays AB, AD and touches ω_1 externally at point T. Prove that T lies on the diagonal AC.

9. (8–9) Let A_M be the midpoint of side BC of an acute-angled triangle ABC, and A_H be the foot of the altitude to this side. Points B_M, B_H, C_M, C_H are defined similarly. Prove that one of the ratios $A_MA_H : A_HA$, $B_MB_H : B_HB$, $C_MC_H : C_HC$ is equal to the sum of two remaining ratios.

10. (8–9) Let N be the midpoint of arc ABC of the circumcircle of triangle ABC, and NP, NT be the tangents to the incircle of this triangle. The lines BP and BT meet the circumcircle for the second time at points P_1 and T_1 respectively. Prove that $PP_1 = TT_1$.

11. (8–9) Morteza marks six points in the plane. He then calculates and writes down the area of every triangle with vertices in these points (20 numbers). Is it possible that all of these numbers are integers, and that they add up to 2019?

12. (8–11) Let $A_1A_2A_3$ be an acute-angled triangle inscribed into a unit circle centered at O. The cevians from A_i passing through O meet the opposite sides at points B_i $(i = 1, 2, 3)$ respectively.

    (a) Find the minimal possible length of the longest of three segments B_iO.

    (b) Find the maximal possible length of the shortest of three segments B_iO.

13. (9–10) Let ABC be an acute-angled triangle with altitude $AT = h$. The line passing through its circumcenter O and incenter I meets the sides AB and AC at points F and N respectively. It is known that $BFNC$ is a cyclic quadrilateral. Find the sum of the distances from the orthocenter of ABC to its vertices.

14. (9–11) Let the side AC of triangle ABC touch the incircle and the corresponding excircle at points K and L respectively. Let P be the projection of the incenter onto the perpendicular bisector of AC. It is known that the tangents to the circumcircle of triangle BKL at K and L meet on the circumcircle of ABC. Prove that the lines AB and BC touch the circumcircle of triangle PKL.

15. (9–11) The incircle ω of triangle ABC touches the sides BC, CA and AB at points D, E and F respectively. The perpendicular from E to DF meets BC at point X, and the perpendicular from F to DE meets BC at point Y. The segment AD meets ω for the second time at point Z. Prove that the circumcircle of the triangle XYZ touches ω.

16. (9–11) Let AH_1 and BH_2 be the altitudes of triangle ABC; let the tangent to the circumcircle of ABC at A meet BC at point S_1, and the tangent at B meet AC at point S_2; let T_1 and T_2 be the midpoints of AS_1 and BS_2 respectively. Prove that T_1T_2, AB and H_1H_2 concur.

17. (10–11) Three circles ω_1, ω_2, ω_3 are given. Let A_0 and A_1 be the common points of ω_1 and ω_2, B_0 and B_1 be the common points of ω_2 and ω_3, C_0 and C_1 be the common points of ω_3 and ω_1. Let $O_{i,j,k}$ be the circumcenter of triangle $A_iB_jC_k$. Prove that the four lines of the form $O_{ijk}O_{1-i,1-j,1-k}$ are concurrent or parallel.

18. (10–11) A quadrilateral $ABCD$ without parallel and without equal sides is circumscribed around a circle centered at I. Let K, L, M and N be the midpoints of AB, BC, CD and DA respectively. It is known that $AB \cdot CD = 4IK \cdot IM$. Prove that $BC \cdot AD = 4IL \cdot IN$.

19. (10–11) Let AL_a, BL_b, CL_c be the bisectors of triangle ABC. The tangents to the circumcircle of ABC at B and C meet at point K_a, points K_b, K_c are defined similarly. Prove that the lines K_aL_a, K_bL_b and K_cL_c concur.

20. (10–11) Let O be the circumcenter of triangle ABC, H be its orthocenter, and M be the midpoint of AB. The line MH meets the line passing through O and parallel to AB at point K lying on the circumcircle of ABC. Let P be the projection of K onto AC. Prove that $PH \parallel BC$.

21. (10–11) An ellipse Γ and its chord AB are given. Find the locus of orthocenters of triangles ABC inscribed into Γ.

22. (10–11) Let AA_0 be the altitude of the isosceles triangle ABC ($AB = AC$). A circle γ centered at the midpoint of AA_0 touches AB and AC. Let X be an arbitrary point of line BC. Prove that the tangents from X to γ cut congruent segments on lines AB and AC.

23. (10–11) In the plane, let a, b be two closed broken lines (possibly self-intersecting), and K, L, M, N be four points. The vertices of a, b and the points K L, M, N are in general position (i.e. no three of these points are collinear, and no three segments between them concur at an interior point). Each of segments KL and MN meets a at an even number of

points, and each of segments LM and NK meets a at an odd number of points. Conversely, each of segments KL and MN meets b at an odd number of points, and each of segments LM and NK meets b at an even number of points. Prove that a and b intersect.

24. (11) Two unit cubes have a common center. Is it always possible to number the vertices of each cube from 1 to 8 so that the distance between each pair of identically numbered vertices would be at most $4/5$? What about at most $13/16$?