XVI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round. Solutions

1. (D.Shvetsov, 8) Let ABC be a triangle with $\angle C = 90^\circ$, and A_0, B_0, C_0 be the midpoints of sides BC, CA, AB respectively. Two regular triangles AB_0C_1 and BA_0C_2 are constructed outside ABC. Find the angle $C_0C_1C_2$.

   Answer. 30°.

   Solution. Since $C_0B_0 = A_0B = A_0C_2$, $C_0A_0 = AB_0 = B_0C_1$ and $\angle C_0A_0C_2 = \angle C_0B_0C_1 = 150^\circ$, we obtain that triangles $C_0A_0C_2$ and $C_1B_0C_0$ are congruent (fig.1). Thus $C_0C_1 = C_0C_2$ and $\angle C_1C_0C_2 = \angle A_0C_0B_0 + \angle B_0C_0C_1 + \angle A_0C_0C_2 = 120^\circ$. Therefore $\angle C_0C_1C_2 = \angle C_0C_2C_1 = 30^\circ$.

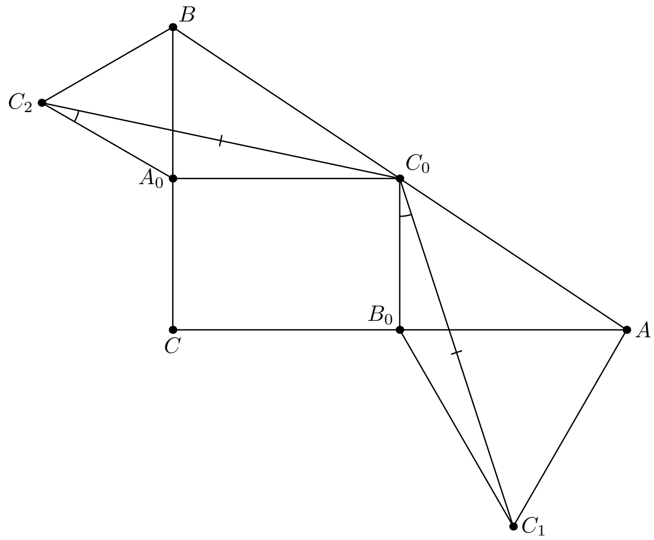

Fig. 1

2. (A.Akopyan, 8) Let $ABCD$ be a cyclic quadrilateral. A circle passing through A and B meets AC and BD at points E and F respectively. The lines AF and BC meet at point P, and the lines BE and AD meet at point Q. Prove that PQ is parallel to CD.

   Solution. Since quadrilaterals $ABCD$ and $ABEF$ are cyclic, we have $\angle CBD = \angle CAD$ and $\angle EBF = \angle EAF$. Thus $\angle PBQ = \angle PAQ$, i.e. $ABPQ$ is also cyclic (fig.2). Therefore CD and PQ are parallel because both lines are antiparallel to AB with respect to lines AP and BQ.

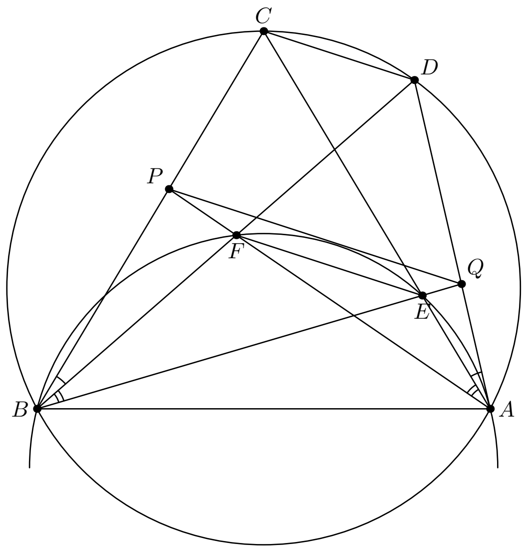

Fig. 2

3. (N.Moskvitin, 8) Let ABC be a triangle with $\angle C = 90°$, and D be a point outside ABC, such that $\angle ADC = \angle BAC$. The segments CD and AB meet at point E. It is known that the distance from E to AC is equal to the circumradius of triangle ADE. Find the angles of triangle ABC.

   Answer. $\angle A = \angle B = 45°$.

   Solution. By the sine law the circumradius of triangle ADE equals to $AE/2\sin\angle ADE$. On the other hand the distance from E to AC equals to $AE\sin\angle BAC$. Then by the assumption we obtain that $2\sin^2\angle A = 1$, i.e. $\angle A = 45°$ (fig.3).

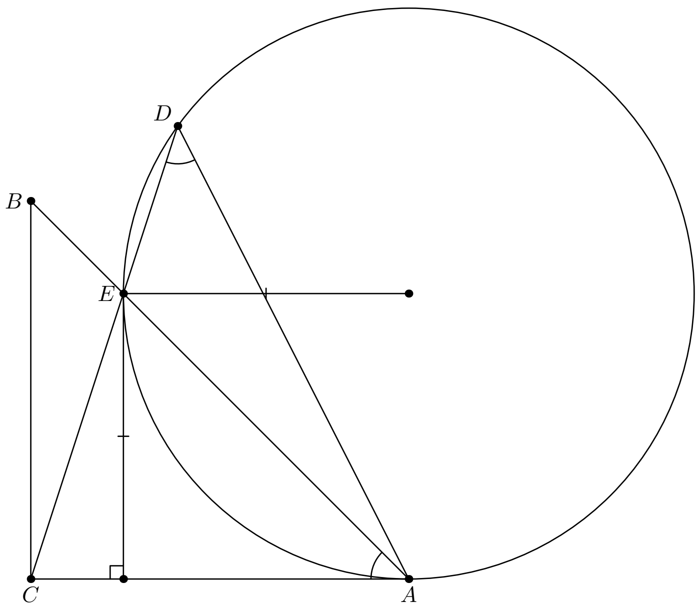

Fig. 3

4. (D.Burek, 8) Let $ABCD$ be an isosceles trapezoid with bases AB and CD. Prove that the centroid of triangle ABD lies on line CF, where F is the projection of D to AB.

   Solution. Let M be the midpoint of AB. Then $FM = CD/2$, therefore the diagonals of trapezoid $CDFM$ divide each other in ratio $2 : 1$ from points C, D (fig.4). Hence the common point of diagonals coincides with the centroid of ABD.

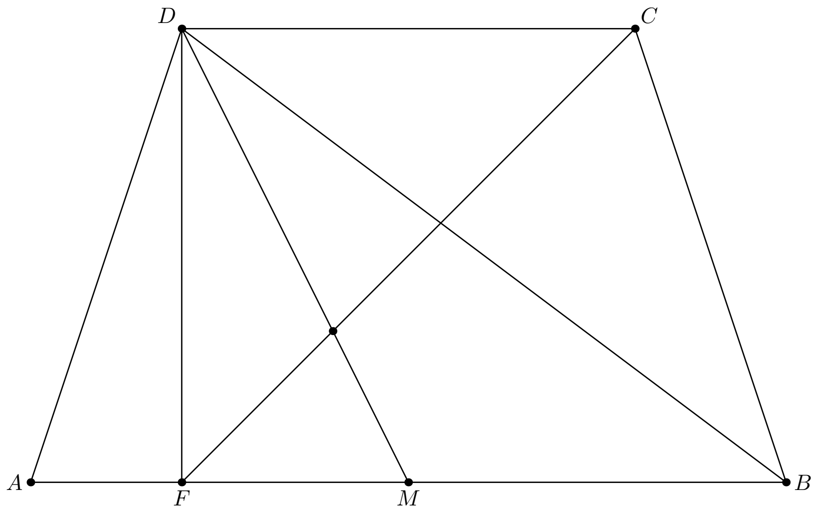

Fig. 4

5. (A.Kulikova, D.Prokopenko, 8–9) Let BB_1, CC_1 be the altitudes of triangle ABC, and AD be the diameter of its circumcircle. The lines BB_1 and DC_1 meet at point E, the lines CC_1 and DB_1 meet at point F. Prove that $\angle CAE = \angle BAF$.

   Solution. Let H be the orthocenter of ABC. Then rays AH and AD are isogonal with respect to angle B_1AC_1. By the isogonal theorem AE and AF are also isogonal.

6. (A.Akopyan, 8–9) Circles ω_1 and ω_2 meet at points P and Q. Let O be the common point of common external tangents to ω_1 and ω_2. A line passing trough O meets ω_1 and ω_2 respectively at points A and B located on the same side with respect to the line PQ. The line PA meets ω_2 for the second time at C, and the line QB meets ω_1 for the second time at D. Prove that O, C, and D are collinear.

   Solution. Since quadrilaterals $ADPQ$ and $BPCQ$ are cyclic, we obtain that $\angle DAC = \angle DQP = \angle BCP$, i.e. $AD \parallel BC$ (fig.6). Since O is the homothety center of the given circles and this homothety maps A to B, it maps D to C.

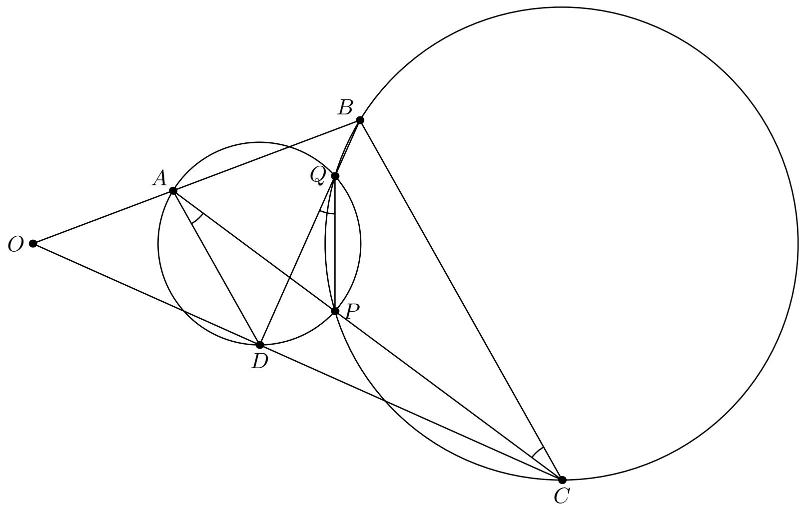

Fig. 6

7. (V.Starodub, 8–9) Prove that the medial lines of triangle ABC meet the sides of triangle formed by its excenters at six concyclic points.

Решение. Let A_b be the projection of A onto the exterior angle bisector through B, and define A_c, B_c, B_a, C_a, and C_b similarly. It is well-known that A_bA_c is the medial line of triangle ABC opposite to A. Thus we want to show that $A_bC_bB_cA_cC_aB_a$ is cyclic.

Let I_a, I_b, and I_c be the excenters opposite to A, B, and C. Then $AA_bI_aA_c$ and BCB_cC_b are both cyclic. Hence $\angle A_cA_bI_a = \angle A_cAI_a = (\pi - \angle B)/2 = \angle CBI_a = \angle C_bB_cI_a$ i.e quadrilateral $A_bA_cB_cC_b$ is cyclic (fig.7). But the perpendicular bisectors to A_cB_c and A_bC_b pass through the midpoints of AB, AC and are parallel to the bisectors of angles C, B respectively. Thus the center of circle $A_bA_cB_cC_b$ coincides with the incenter of the medial triangle. This yields that B_a, C_a lie on the same circle.

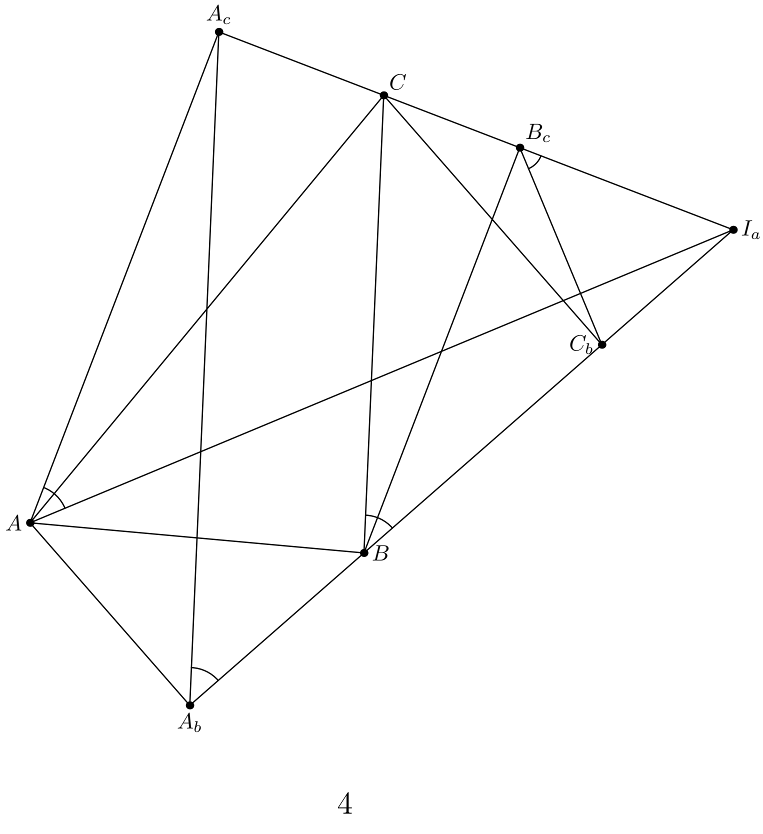

Fig. 7

8. (P.Ryabov, 8–9) Two circles meeting at points P and R are given. Let l_1, l_2 be two lines passing through P. The line l_1 meets the circles for the second time at points A_1 and B_1. The tangents at these points to the circumcircle of triangle A_1RB_1 meet at point C_1. The line C_1R meets A_1B_1 at point D_1. Points A_2, B_2, C_2, D_2 are defined similarly. Prove that the circles D_1D_2P and C_1C_2R touch.

   Solution. Let us prove that these circles touch at R. Note that D_1, D_2, P, and R are concyclic because D_1R and D_2R are corresponding lines in similar triangles A_1RB_1 and A_2RB_2. Let the tangents to the circles at A_1 and A_2 meet at point X, and the tangents at B_1 and B_2 meet at Y. Note that $\angle A_1XA_2 = \angle A_1RA_2$ (the rotation angle), therefore A_1, X, R, and A_2 are concyclic. Similarly X, R, C_1, C_2, and Y are concyclic. Now we have to prove that $D_1D_2 \parallel C_1C_2$. We have $\angle D_1D_2R = \angle D_1PR = \angle RXC_1 = \angle RC_2C_1$, therefore these lines are parallel, q.e.d.

9. (G.Filippovsky, 8–9) The vertex A, the circumcenter O, and the Euler line l of triangle ABC are given. It is known that l meets AB and AC at two points equidistant from A. Restore the triangle.

   Solution. We have that the Euler line is parallel to the exterior angle bisector at A. Since AO and AH are isogonal rays with respect to $\angle A$, it follows that $AO = AH$. Thus we can recover H as the second point where the circle with center A and radius AO meets the Euler line. Furthermore let line AH meet the circumcircle (which we can recover because we know its center O and one point on it, namely A) again at D. Then B and C are the points where the perpendicular bisector of segment HD meets the circumcircle.

   Remark. Since AH is twice the distance from O to BC in each triangle and AH equals the circumradius in our triangle, we have that $\angle A = 60^\circ$ or $\angle A = 120^\circ$. It is not too difficult to show that if $\angle A = 60^\circ$ then the Euler line is parallel to the exterior angle bisector at A, and if $\angle A = 120^\circ$ then it is parallel to the interior angle bisector at A. Thus in the problem we must necessarily have that $\angle A = 60^\circ$.

10. (A.Ivanischuk, 8–9) Given are a closed broken line $A_1A_2 \dots A_n$ and a circle ω which touches each of lines $A_1A_2, A_2A_3, \dots, A_nA_1$. Call the link *good*, if it touches ω, and *bad* otherwise (i.e. if the extension of this link touches ω). Prove that the number of bad links is even.

    Solution. Let O be the center of the circle. For all i, let T_i be the tangency point of the circle and line A_iA_{i+1}. (We define A_{n+1} to be simply another name for point A_1.) We say that triangle ABC is *positively oriented* if vertices A, B, and C occur in counterclockwise order along the boundary of the triangle, and we say that it is *negatively oriented* otherwise.

    Note that triangles OA_iT_i and $OA_{i+1}T_i$ are identically oriented if and only if segment A_iA_{i+1} is bad. On the other hand, triangles $OA_{i+1}T_i$ and $OA_{i+1}T_{i+1}$ are always differently oriented. Therefore, segment A_iA_{i+1} is bad if and only if triangles OA_iT_i and $OA_{i+1}T_{i+1}$ are oriented differently. Thus the number of bad segments equals the number of changes of orientation in the cyclic sequence of triangles $OA_1T_1, OA_2T_2, \dots, OA_nT_n$, and so it must be even. (Since orientation must change an even number of times in order to turn out the same when we arrive back at the beginning of the sequence.)

11. (A.Utkin, 8–9) Let ABC be a triangle with $\angle A = 60^\circ$, AD be its bisector, and PDQ be a regular triangle with altitude DA. The lines PB and QC meet at point K. Prove that AK is a symmedian of ABC.

    Solution. Let us prove that P and B lie on the different sides with respect to AD. In fact, in the other case let U be the common point of AD and PD, and V be the common point of AC and QD. Then $AUDV$ is a rhombus, because $\angle UAD = \angle VAD = \angle UDA = \angle VDA = 30^\circ$. Applying the Pappus theorem to points (P, A, Q) and (B, D, C) we obtain that $PB \parallel QC$, which contradicts the assumption (fig.11).

    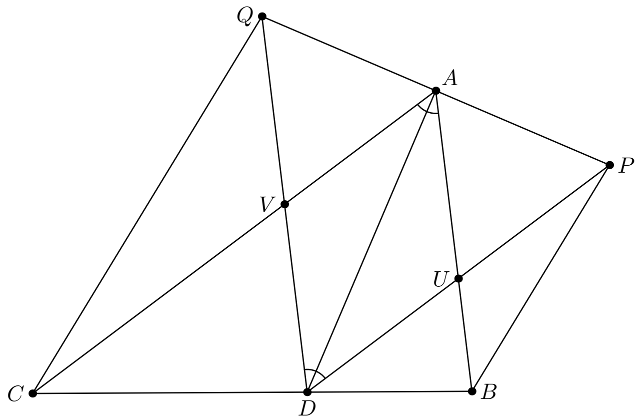

    Fig. 11

    Now note that by the isogonal theorem PB and QC, PC and QB are isogonal with respect to angle A. But as shown above PC and QB are parallel, and since $AP = AQ$, they are parallel to the median of ABC. This yields the required assertion.

12. (A.Mudgal, P.Srivastava, 8–10) Let H be the orthocenter of a nonisosceles triangle ABC. The bisector of angle BHC meets AB and AC at points P and Q respectively. The perpendiculars to AB and AC from P and Q meet at K. Prove that KH bisects the segment BC.

    Solution. Note that A is the orthocenter of triangle BHC. Therefore, the problem remains the same when we swap A and H. Reformulate it as follows.

    Let H be the orthocenter of nonisosceles triangle ABC. The interior angle bisector through A meets lines BH and CH at points P and Q, respectively. The perpendiculars to BH and CH at P and Q, respectively, meet at K. Prove that line AK bisects segment BC.

    Let M, S, and T be the midpoints of chord BC, arc BAC, and arc BC of circle ABC respectively. Then point T lies on line APQ, and we want to show that points A, K, and M are collinear.

    Furthermore, let L be the intersection point of lines KH and PQ, let N be the projection of point M onto line $APQT$, and let R be the intersection point of lines AH and MN (fig.12).

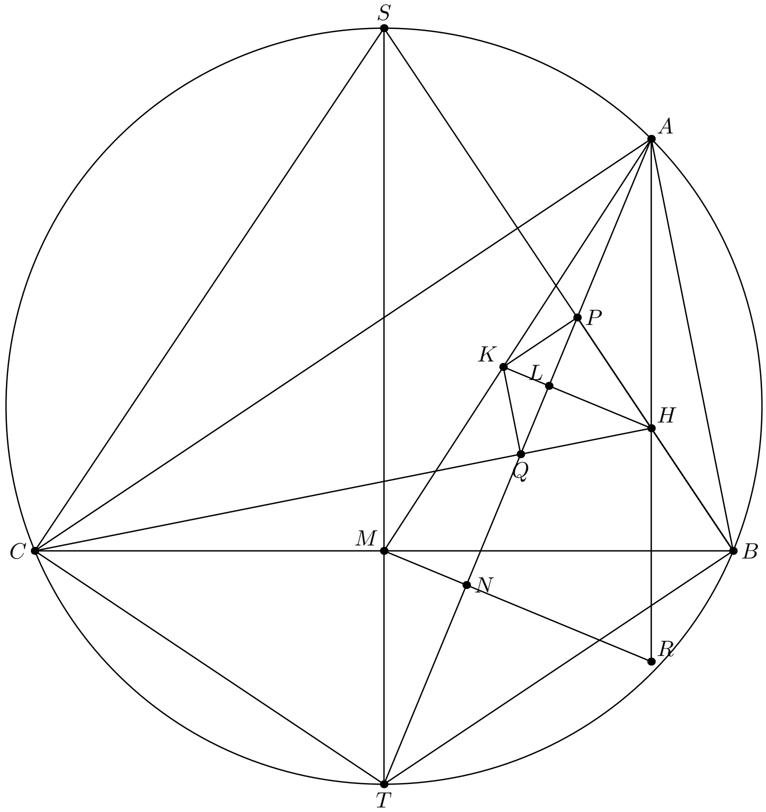

Fig. 12

By a simple angle chase, figures $HKLPQ$ and $STMBC$ are similar deltoids. Therefore, $HL : LK = SM : MT$. Since lines AS and MN are parallel, we have also $SM : MT = AN : NT$. Finally, since lines AHR and SMT are parallel as well, $AN : NT = RN : NM$. Now, AHR and ALN being straight lines, lines HLK and RNM being parallel, and $HL : LK = RN : NM$, we obtain that points A, K, and M are collinear, as needed.

13. (A.Utkin, 9–11) Let I be the incenter of triangle ABC. The excircle with center I_A touches the side BC at A'. The line l passing through I and perpendicular to BI meets I_AA' at point K lying on the medial line parallel to BC. Prove that $\angle B \leq 60^\circ$.

    Solution. Let AH_A be the altitude of triangle, M be the midpoint of this altitude, and N be the common point of AH_A and BI. Then A', I, M — the projections of K to BC, BI, AH_A respectively — are collinear, therefore, $BKNH_A$ is a cyclic quadrilateral and $\angle BKH_A = \angle BNH_A = 90^\circ - \angle B/2$.

    Since the midpoint M_C of AB is equidistant from B and H_A, and $M_CK \parallel BH_A$, we obtain that $\angle BKH_A < \angle BM_CH_A = 180^\circ - 2\angle B$, this yields the required inequality.

14. (F.Ivlev, 9–11) A nonisosceles triangle is given. Prove that one of the circles touching internally its incircle and circumcircle and touching externally one of its excircles passes through a vertex of the triangle.

    Solution. Let ω be the incircle and let ω_A be the excircle opposite to A. Also let t be the second common internal tangent of ω and ω_A.

    Consider the inversion with center A which swaps ω and ω_A. This inversion maps line t onto a circle s passing through A which is tangent to ω internally and to ω_A externally, and whose tangent at A is parallel to t.

On the other hand, lines BC and t are symmetric with respect to interior angle bisector of angle A, and so the tangents to s and the circumcircle of ABC at A coincide. Thus s is also tangent to the circumcircle (internally), and we are done.

15. (A.Akopyan, 9–11) A circle passing through the vertices B and D of quadrilateral $ABCD$ meets AB, BC, CD, and DA at points K, L, M, and N respectively. A circle passing through K and M meets AC at P and Q. Prove that L, N, P, and Q are concyclic.

    Solution. By Pascal's theorem for cyclic hexagon $BKMDNL$, lines KM and LN meet at some point X on line AC. In circle $BDKLMN$ we have $KX \cdot XM = LX \cdot XN$ (fig.15). Then in circle $KMPQ$ we have $KX \cdot XM = PX \cdot XQ$. The claim follows.

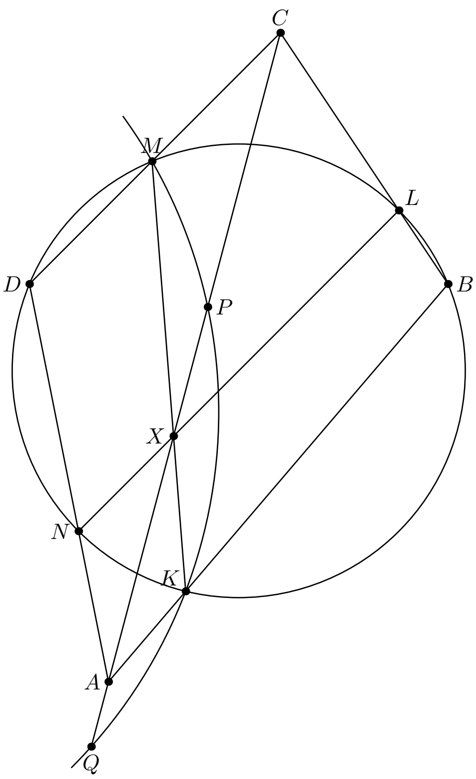

Fig. 15

16. (P.Ryabov, 9–11) Cevians AP and AQ of a triangle ABC are symmetric with respect to its bisector. Let X, Y be the projections of B to AP and AQ respectively, and N, M be the projections of C to AP and AQ respectively. Prove that XM and NY meet on BC.

    Solution. Note that M, N, X, and Y lie on the circle Ω. In fact the similarity of triangles ABX and ACM yields that $AX : AM = AB : AC$. Similarly $AN : AY = AC : AB$. Thus $AX \cdot AN = AY \cdot AM$. Also, since the perpendicular bisectors to XN and YM pass through the midpoint T of BC, we obtain that T is the center of Ω.

    Let AH be the altitude of ABC and Z be the common point of MN and XY. Then Z lies on AH because AH, MN and XY are radical axes of circles Ω, $ABXY$ and $ACMN$.

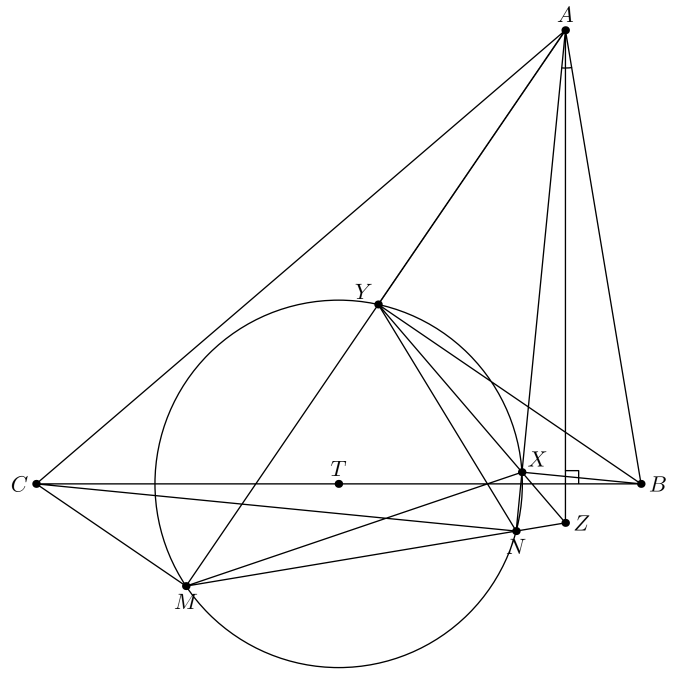

Fig. 16

Finally, let MX and NY meet at W. Then W is the pole of line AZ with respect to circle $XMYN$, therefore, $AZ \perp TW$, i.e. W lies on BC.

Remark. After the correspondence round was published we learned that this problem is the same as Balkan Mathematical Olympiad 2019 problem 3.

17. (A.Kazakov, 10–11) Chords A_1A_2 and B_1B_2 meet at point D. Suppose D' is the inversion image of D and the line A_1B_1 meets the perpendicular bisector to DD' at a point C. Prove that $CD \parallel A_2B_2$.

    Solution. Since C lies on the radical axis of the given circle and the point D, we have $CD^2 = CB_1 \cdot CA_1$, therefore, $\angle CDB_1 = \angle DA_1C = \angle A_2B_2D$.

18. (D.Shvetsov, Yu.Zaytseva, 10–11) Bisectors AA_1, BB_1, and CC_1 of triangle ABC meet at point I. The perpendicular bisector to BB_1 meets AA_1, CC_1 at points A_0, C_0 respectively. Prove that the circumcircles of triangles A_0IC_0 and ABC touch.

    First solution. The perpendicular bisector to BB_1 and the bisector of angle A meet on the circumcircle of triangle ABB_1, therefore, $\angle IBA_0 = \angle IAB$. Similarly $\angle IBC_0 = \angle ICB$. Then $\angle A_0BC_0 = \angle A_1IC$, i.e. I, A_0, C_0, and B are concyclic (fig.18). The angle between the tangent to this circle at B and the line BB_1 equals to $\angle BC_0A_0 + \angle A_0BI = \angle IAC + \angle AIB_1 = \angle BB_1C$. The angle between BB_1 and the tangent to the circumcircle of ABC is the same. Thus both circles touch at B.

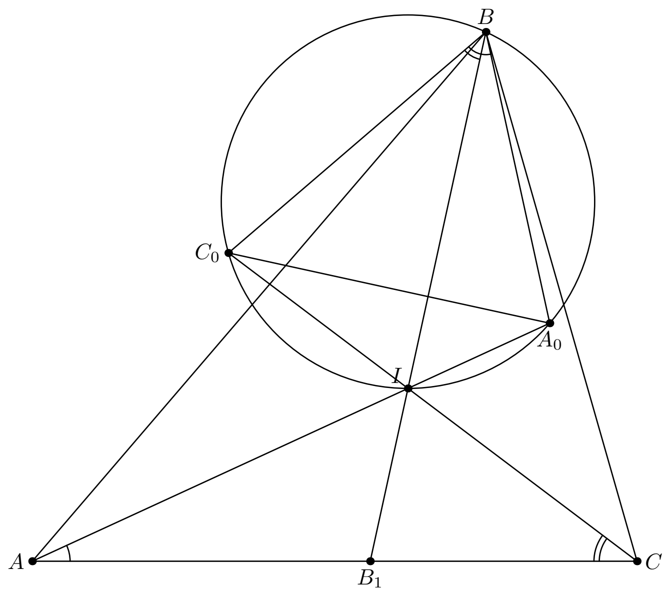

Fig. 18

Second solution. Let angle bisectors AA_1, BB_1, and CC_1 meet the circumcircle of triangle ABC again at A', B', and C' respectively. Also let M be the midpoint of BB_1 and N be the midpoint of BI. Note that line $A'C'$ is the perpendicular bisector of segment BI.

We claim that quadrilaterals A_0IC_0B and $A'BC'B'$ are similar. Once we have this, the rest of the problem follows easily: Since $A'BC'B'$ is cyclic, circle A_0IC_0 passes through B. Also, since lines A_0C_0 and $A'C'$ are parallel, circles A_0IC_0 and $A'IC'$ are tangent at I. But now reflection about line $A'C'$ preserves the first circle (since it passes through B and I), and it maps the second circle onto the circumcircle of triangle ABC. Therefore, circles A_0IC_0 and ABC are tangent at B.

To prove the similarity, note that triangles A_0IC_0 and $A'BC'$ are similar by equal angles, and segments IM and BN are corresponding altitudes in them. After that, we are only left to prove $BM : BI = B'N : B'B$. This is equivalent to $BB_1 : BI = (B'I + B'B) : B'B$. After subtraction of 1 from both sides, this becomes $IB_1 : BI = B'I : B'B$. Now $IB_1 : BI = AB_1 : AB = B'C : B'B = B'I : B'B$. The solution is complete.

19. (A.Zaslavsky, 10–11) Quadrilateral $ABCD$ is such that $AB \perp CD$ and $AD \perp BC$. Prove that there exists a point such that the distances from it to the sidelines are proportional to the lengths of the corresponding sides.

Решение. By assumption, altitudes AA' and CC' of triangle ABC meet at D. Reflect the median of ABC from B about the corresponding bisector and find the common point L of the obtained line with $A'C'$. Since A, C, A', C' lie on the circle with diameter AC, we have $\angle LA'C = \angle DAC = \pi/2 - \angle BCA$. Hence the ratio of distances from L to AB and CD equals $\sin\angle LA'B/\sin\angle LA'C = \operatorname{tg}\angle BCA$. But $AB = 2R\sin\angle BCA$, $CD = 2R\cos\angle BCA$, where R is the circumradius of ABC. Therefore the distances from L to AB and CD are proportional to their sidelengths. Similarly for sides AD and BC. Also since L lies on the symmedian of ABC, the distances from L to AB and BC are also proportional to their sidelengths, i.e. L is the required point.

20. (M.Didin, 10–11) The line touching the incircle of triangle ABC and parallel to BC meets the external bisector of angle A at point X. Let Y be the midpoint of arc BAC of the circumcircle. Prove that the angle XIY is right.

First solution. Let the tangent parallel to BC touch the incircle at D. Also let the incircle touch sides BC, CA, and AB at A', B', and C', respectively. Let M be the midpoint of BC. Finally suppose without loss of generality that $AB > AC$, and let Z and T be the projections of Y onto lines AB and IA', respectively.

Triangles AYZ and IAB' are similar because $\angle AYZ = \angle IAB' = \angle A/2$ and $\angle AZY = \angle IB'A = 90°$. Thus $AY : AZ = IA : IB'$. On the other hand, by the Archimedes lemma, Z is the midpoint of the broken line ABC. Thus $AZ = (c - b)/2 = A'M = YT$. Also $IB' = ID$. Therefore, $AY : YT = AI : ID$ (fig.20).

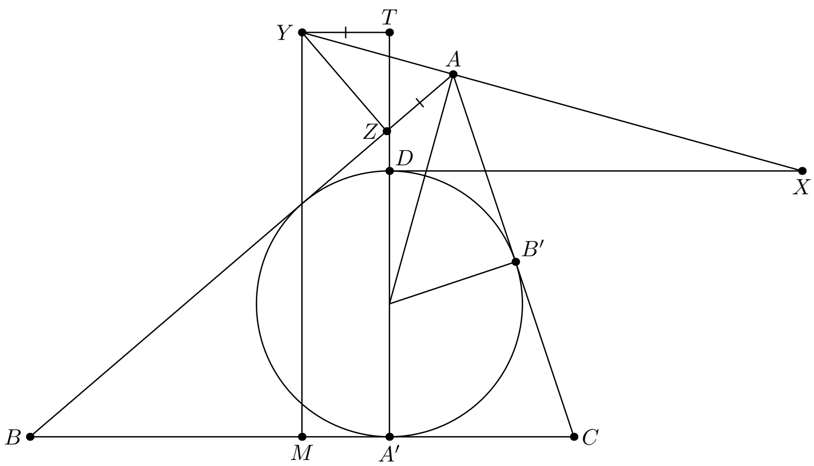

Fig. 20

Since also $AY \perp AI$ and $YT \perp ID$, there exists a spiral similarity with center A and angle $90°$ which maps Y onto I, T onto D, line AI onto line AX, and line TI onto line DX. Therefore, the same spiral similarity maps I onto X, and so also line YI onto line IX. The solution is complete.

Second solution. Since DX is the polar of D with respect to the incircle, and AX is the polar of the midpoint A_0 of $B'C'$, we have to prove that $DA_0 \parallel IY$. Note that the triangle $A'B'C'$ is homothetic to the triangle $I_aI_bI_c$ formed by the excenters, this homothety maps Y to A_0, and I to the orthocenter of $A'B'C'$. But D, the point opposite to A', is the reflection of the orthocenter about A_0, q.e.d.

21. (A.Zaslavsky, 10–11) The diagonals of bicentric quadrilateral $ABCD$ meet at point L. Given are three segments equal to AL, BL, CL. Restore the quadrilateral using a compass and a ruler.

Solution. Since $ABCD$ is cyclic, we have $AL \cdot LC = BL \cdot LD$. So we can recover the length of DL. Let $AL = a$, $BL = b$, $CL = c$, and $DL = d$.

Let the incircle of $ABCD$ touch segments AB, BC, CD, and DA at points P, Q, R, and S, respectively. It is well-known that in a circumscribed quadrilateral lines PR and QS meet at L. Moreover since $ABCD$ is cyclic, we have that lines PR and QS are in fact the two angle bisectors of the angle formed by lines AC and BD.

Let $AS = AP = a'$, $BP = BQ = b'$, $CQ = CR = c'$, and $DR = DS = d'$. By the angle bisector theorem for triangle ALB, we have that $AL : LB = AP : PB$ or, equivalently, $a' : a = b' : b$. Analogously, $a' : a = b' : b = c' : c = d' : d$. Denote this ratio by x. Then $AB = (a+b)x$, and analogously for BC, CD, and DA.

By Ptolemy's theorem for $ABCD$, we have that $AB \cdot CD + BC \cdot DA = AC \cdot BD$. That is,

$$x^2(a+b)(c+d) + x^2(b+c)(d+a) = (a+c)(b+d).$$

Therefore,

$$x = \sqrt{(a+c)(b+d)/((a+b)(c+d)+(b+c)(d+a))}.$$

Using this value of x, we can construct segments AB, BC, CD, DA, and so the quadrilateral $ABCD$.

Remark. We can also express the lengths of segments LP, LQ, LR, LS through a, b, c, d and $\varphi = \angle ALB$ and find $\cos\varphi$ from equality $PL \cdot LR = QL \cdot LS$.

22. (A.Khurmi, K.V.Sudharshan, 10–11) Let Ω be the circumcircle of cyclic quadrilateral $ABCD$. Consider such pairs of points P, Q of diagonal AC that the rays BP and BQ are symmetric with respect the bisector of angle B. Find the locus of circumcenters of triangles PDQ.

    Solution. Let lines BP and BQ meet circle $ABCD$ again at points R and S, respectively. Since $\angle ABP = \angle CBQ$, we have that $\smile AR = \smile CS$. Thus lines AC and RS are parallel. Consequently, a homothety centered at B maps triangle BPQ onto triangle BRS, and so circles BPQ and $BRS = ABCD$ are tangent at point B.

    Let the tangent to circle $ABCD$ at B meet line AC at point X, and let line DX meet circle $ABCD$ again at E. Note that both points X and E are fixed when points P and Q vary.

    Then line BX is the radical axis of circles $ABCD$ and BPQ, whereas line AC is the radical axis of circles BPQ and DPQ. Consequently, line DEX is the radical axis of circles $ABCD$ and DPQ. Thus point E always lies on circle DPQ when points P and Q vary (fig.22). Therefore, the locus of the circumcenters of all triangles DPQ is the perpendicular bisector of segment DE, except all points O on this perpendicular bisector for which the circle with center O through D and E does not intersect line AC. (All such exceptional points O form an open interval.)

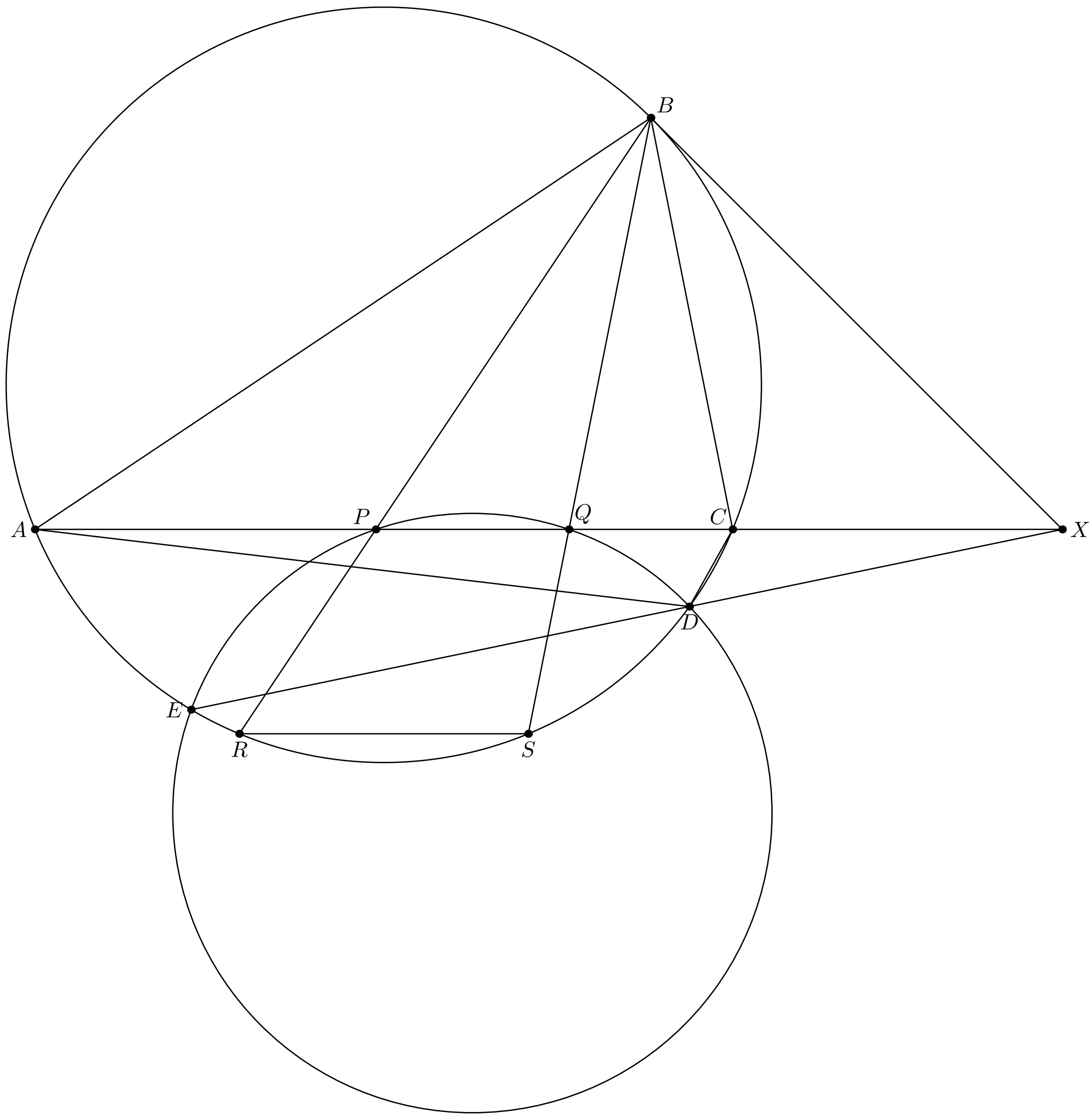

Fig. 22

23. (N.Beluhov, 10–11) A non-self-intersecting polygon is *nearly convex* if precisely one of its interior angles is greater than $180°$.

One million distinct points lie in the plane in such a way that no three of them are collinear. We would like to construct a nearly convex one-million-gon whose vertices are precisely the one million given points. Is it possible that there exist precisely ten such polygons?

Answer. No, it is not.

Solution. Let $P_1, P_2, \ldots, P_n$ be all given points (with $n = 1000000$) and let $H = H_1H_2 \ldots H_k$ be their convex hull. We refer to points $H_1, H_2, \ldots, H_k$ as *outer* points, and to the remaining $n - k$ points as *inner* points.

Let $Q_1, Q_2, \ldots, Q_n$ be a permutation of $P_1, P_2, \ldots, P_n$ such that polygon $Q = Q_1Q_2 \ldots Q_n$ is nearly convex. All sides of H except precisely one, say s, must also be sides of Q.

Let R be the unique vertex of Q such that the interior angle of Q at R is greater than

$180°$. Then R is an inner point and all inner points other than R lie in the interior of the triangle spanned by R and s.

If there is a unique inner point, then it must coincide with R. In that case, every side of H may play the role of s, and we have $n-1$ options for Q. Since $1000000-1 > 10$, this case cannot occur.

If there are precisely two inner points R_1 and R_2, then any one of them may play the role of R. Suppose, for instance, that $R \equiv R_1$. Then s is necessarily that side of H where the ray $R_1R_2^{\rightarrow}$ pierces the contour of H, say H_uH_{u+1}, and Q may have precisely the forms $H_1H_2 \dots H_uR_1R_2H_{u+1} \dots H_k$ and $H_1H_2 \dots H_uR_2R_1H_{u+1} \dots H_k$. Consequently, in this case we have precisely four options for Q and so this case is ruled out as well.

We are left to consider the case when there are at least three inner points. Let $G = G_1G_2 \dots G_m$ be their convex hull, $m \geq 3$.

We say that a vertex G_i of G is *promising* if the rays $G_iG_{i-1}^{\rightarrow}$ and $G_iG_{i+1}^{\rightarrow}$ meet the same side of H. (Here $G_0 \equiv G_m$ and $G_{m+1} \equiv G_1$.) Any G_i that can play the role of R must be promising. However, perhaps not all promising G_i can play the role of R.

We are going to show that there are at most three promising vertices of G.

To this end suppose by way of contradiction that there are at least four promising vertices of G. Pick any four of them and label them A, B, C, and D in such a way that $ABCD$ is a convex quadrilateral and $\angle A + \angle B \geq 180°$.

Then ray $BC^{\rightarrow}$ lies in the interior of the angle spanned by rays $AB^{\rightarrow}$ and $AD^{\rightarrow}$. Since A is promising, it follows that all three rays meet the same side of H.

However, since B is also promising, rays $BA^{\rightarrow}$ and $BC^{\rightarrow}$ meet the same side of H as well. It follows that rays $AB^{\rightarrow}$ and $BA^{\rightarrow}$ meet the same side of H. We have arrived at a contradiction.

Let then G_l be a promising vertex of G with rays $G_lG_{l-1}^{\rightarrow}$ and $G_lG_{l+1}^{\rightarrow}$ meeting the same side H_uH_{u+1} of H.

Consider a ray $r^{\rightarrow}$ emanating from G_l and pointing in the direction of ray $G_lG_{l-1}^{\rightarrow}$. Rotate $r^{\rightarrow}$ about G_l so that it remains in the interior of $\angle G_{l-1}G_lG_{l+1}$ until it takes the direction of ray $G_lG_{l+1}^{\rightarrow}$. Let $J_1 \equiv G_{l-1}, J_2, \dots, J_{n-k-1} \equiv G_{l+1}$ be all inner points other than G_l in the order in which they are swept by $r^{\rightarrow}$. Let also $J_0 \equiv H_u$ and $J_{n-k} \equiv H_{u+1}$. Then for some $0 \leq v \leq n-k-1$ we must have

$$Q \equiv H_1H_2 \dots H_{u-1}J_0J_1 \dots J_vG_lJ_{v+1}J_{v+2} \dots J_{n-k}H_{u+2}H_{u+3} \dots H_k.$$

Consider the non-selfintersecting polygon

$$Q' = H_1H_2 \dots H_{u-1}J_0J_1 \dots J_{n-k}H_{u+2}H_{u+3} \dots H_k.$$

Since Q' is nonconvex, at least one of its interior angles must be greater than $180°$. Since its interior angles at H_1, H_2,…,$H_u \equiv J_0$, $H_{u+1} \equiv J_{n-k}$, …, H_k are all smaller than $180°$, there must exist some $1 \leq w \leq n-k-1$ such that the interior angle of Q' at J_w is greater than $180°$.

It follows that we must have either $v = w-1$ or $v = w$. (Otherwise Q would have two interior angles greater than $180°$, one at G_l and one at J_w.) Thus each promising vertex G_l of G yields at most two options for Q.

Consequently, the total number of options for Q does not exceed three promising vertices G_l of G times two options for Q associated with each such vertex. Since $3 \cdot 2 = 6 < 10$, it is impossible that there exist precisely seven options for Q.

24. (I.Bogdanov, 11) Let I be the incenter of a tetrahedron $ABCD$, and J be the center of the exsphere touching the face BCD and the planes containing three remaining faces (outside these faces). The segment IJ meets the circumsphere of the tetrahedron at point K. Which of two segments IK and JK is longer?

    Answer. IK.

    Solution. Consider the plane passing through line AIJ and perpendicular to the plane BCD. It intersects both spheres by their great circles. Let the tangents from A to these circles meet BCD at points X and Y. Then I and J are the incenter and the excenter of triangle AXY, thus the midpoint of IJ lies on the arc XY of the circumcircle of this triangle. But X and Y lie inside the circumsphere of the tetrahedron, therefore arc XY also lies inside it and $IK > IJ/2 > JK$.