XXI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. First day. 8 grade. Solutions
July 31, 2025.

1. (I.Kukharchuk, E.Galakhova.) A cyclic pentagon $ABCDE$ is given. The diagonals AC and CE are equal and meet BD at points M and N respectively. It is known that $BM = ND$, $BC \neq CD$. Prove that the reflection of C about the midpoint of BD lies on AE.

   Solution. From the assumption we have $CM \cdot MA = BM \cdot MD = DN \cdot BN = CN \cdot NE$. Hence $CM = CN$ or $CM = EN$. The first case is impossible because $BC \neq CD$, in the second case the midpoint of MN lies on the medial line of triangle ACE (fig. 8.1), which is equivalent to the required assertion.

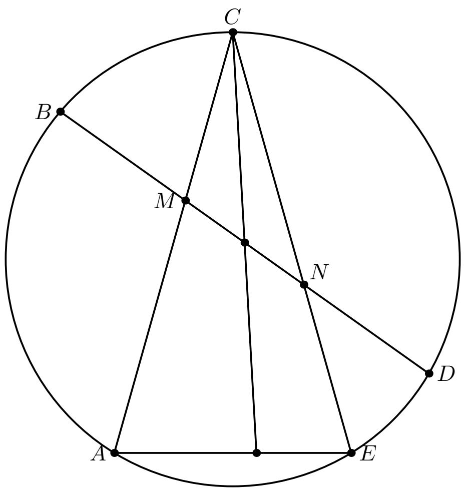

Fig. 8.1.

2. (L.Emelyanov) Let CH be an altitude of triangle ABC; and CA', CB' be bisectors of triangles CBH, CAH respectively. Prove that the circumcenter of triangle $CA'B'$ coincides with the incenter of triangle ABC if and only if $\angle ACB = 90°$.

   Solution. Let the incenter I of triangle ABC coincide with the circumcenter of triangle $A'B'C$. Then it lies on the circumcircle of triangle $A'BC$ as the common point of the bisector of angle B and the perpendicular bisector to $A'C$. Therefore $\angle CIB = \angle CA'B$. Similarly $\angle CIA = \angle CB'A$ (fig. 8.2). Thus $\angle AIB = 180° - \angle A'CB'$. On the other hand $\angle AIB = 90° + \angle A'CB'$, which yields $\angle C = 2\angle A'CB' = 90°$. Similarly we obtain the converse.

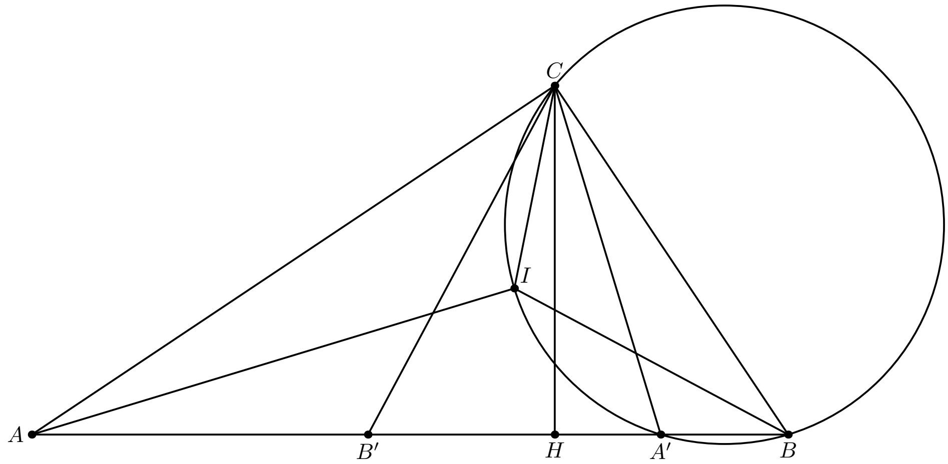

Fig. 8.2.

3. (F.Nilov) Can we choose more than six points on the plane not on a single line, and color them into three colors so that any line through two marked points of different colors contains exactly one more marked point, of the remaining color?

Answer. Yes, we can.

Solution. Let points A_1, B_1, C_1 lie on a line ℓ_1, points A_2, B_2, C_2 lie on a line ℓ_2, the lines A_1B_2 and A_2B_1 meet at point C_3, the lines A_1C_2 and A_2C_1 meet at point B_3, the lines B_1C_2 and B_2C_1 meet at point A_3. Then by the Pappus theorem A_3, B_3, C_3 are collinear (fig. 8.3). Now coloring A_1, A_2, A_3 into the first color, B_1, B_2, B_3 into the second color, and C_1, C_2, C_3 into the third one we obtain the required configuration.

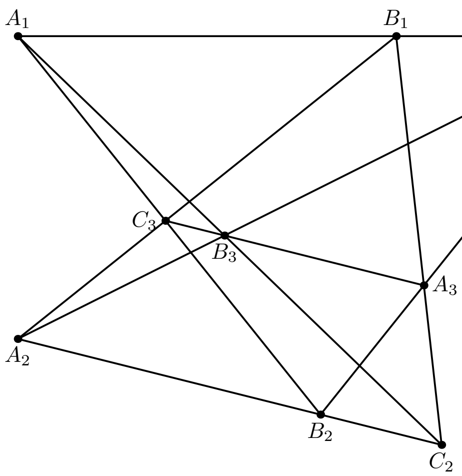

Fig. 8.3.

Remark. Using the addition of points on a cubic we can construct an example with an arbitrary great number of points: take a point P such that

$3nP = 0$ and color all points $3kP$ into the first color, all points $(3k+1)P$ into the second one, and all points $(3k-1)P$ into the third color.

4. (L.Shatunov) Let AA_1 and CC_1 be bisectors of a triangle ABC, and B_0 be the midpoint of the arc AC on the circumcircle of $\triangle ABC$, not containing B. The circumcircles of triangles AA_1B_0 and CC_1B_0 meet the lines BC and AB at points P and Q respectively. Prove that the incenter of $\triangle ABC$ lies on PQ.

   First solution. Let the line passing through I and parallel to AC meet BC at point P'. Then $\angle P'IC = \angle ICA = \angle ICP'$, therefore $IP' = P'C$. On the other hand $IB_0 = B_0C$, thus the triangles $P'IB_0$ and $P'CB_0$ are congruent, i.e. $\angle B_0P'C = (180° - \angle C)/2 = \angle IAB_0 = \angle CPB_0$, and P' coincides with P. Similarly $IQ \parallel AC$ (fig. 8.4).

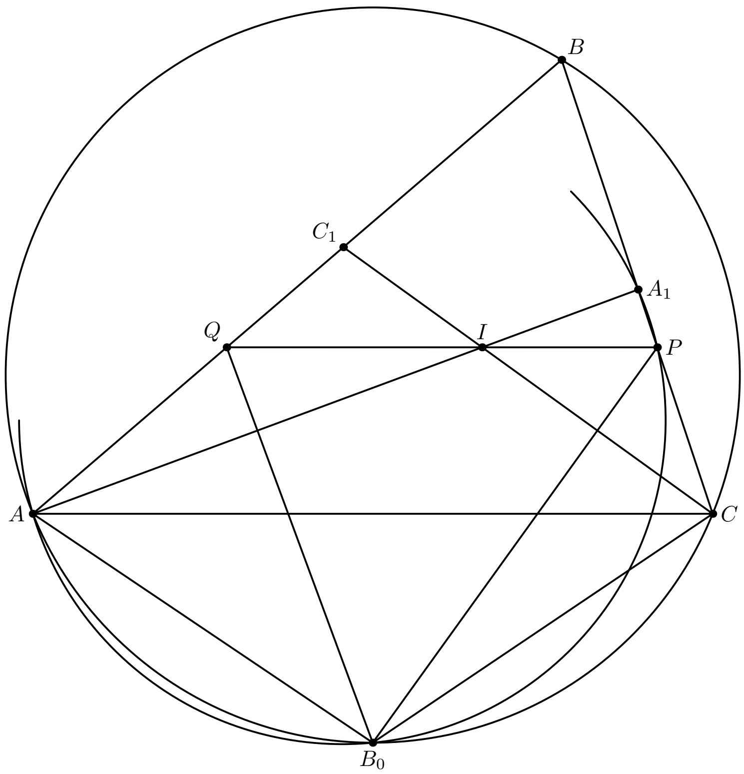

Fig. 8.4.

   Second solution. Let A_0, C_0 be the midpoints of arcs BC, AB; and P' be the common point of A_0B_0 and BC. Then $AA_1P'B_0$ is a cyclic quadrilateral, because $\angle AB_0A_0 = \angle BA_1A$, thus P' coincides with P. Similarly Q lies on B_0C_0. Applying the Pascal theorem to the hexagon $ABCC_0B_0A_0$ we obtain the required.

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Final round. Second day. 8 grade. Solutions

August 1, 2025.

5. (M.Volchkevich) The distance from the vertex of the right angle of a right-angled triangle to the bisector of its acute angle equals a quarter of its hypotenuse. Find all possible values of the angles of this triangle.

   Answer. 60° and 30°, or 36° and 54°.

   Solution. Let M be the midpoint of the hypotenuse AB of triangle ABC, and L be the reflection of C about the bisector of angle A. Then $CL = AB/2 = CM$, and two cases are possible.

   1. Points M and L coincide. Then the bisector of angle A of triangle ACM coincides with the altitude of this triangle, therefore $AC = AM = CM$, and $\angle A = 60^\circ$.

   2. Points M and L are different. Then $AC = AL$, and $LC = CM = MA$. Therefore $\angle ALC = \angle LMC = 2\angle A$ (fig. 8.5). On the other hand $2\angle ALC + \angle A = 180^\circ$. Thus $\angle A = 36^\circ$.

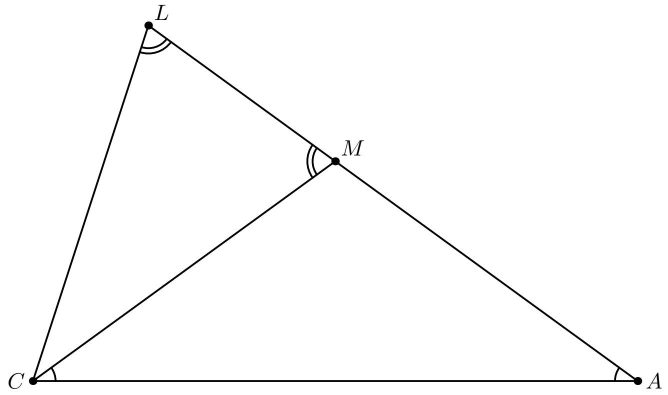

Fig. 8.5.

6. (I.Mikhailov) Let $ABCD$ be a convex quadrilateral with $\angle ABD = \angle ACD = 90^\circ$. Two circles with diameters AB and CD meet at points P and Q. Prove that $2PQ < AD$.

   Solution. Let K, L be the projections of B, C to AD. Then the circles with diameters AB, CD pass through K and L respectively, and their common points P, Q lie on the arcs BK and CL (fig. 8.6). Since $\angle BAK < 90^\circ$, we obtain that $PQ \le BK \le AD/2$, an equality is possible only when B and C coincide.

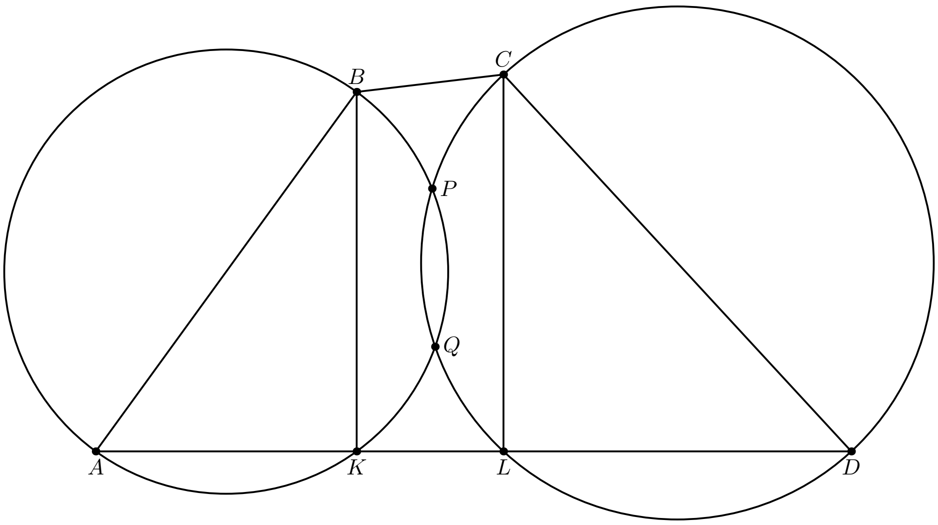

Fig. 8.6.

7. (K.Belsky) A regular triangle ABC is inscribed into a circle Ω. Circles Ω_A, Ω_B, Ω_C centered at A, B, C respectively pass through a point P lying on Ω and have a common tangent. Prove that there exists a line touching two of these circles and passing through some vertex of ABC.

   First solution. Let P lie on the arc AB of Ω. Then by the Pompeiu theorem $PC = PA+PB$, hence the length of a common tangent to Ω_A and Ω_C equals $\sqrt{AC^2-(PC-PA)^2} = \sqrt{BC^2-PB^2}$, i.e. the length of a tangent from C to Ω_B. Similarly the length of a common tangent to Ω_B and Ω_C equals to the length of a tangent from C to Ω_A. Also, since Ω_A, Ω_B, Ω_C have a common tangent, the length of a common tangent to one pair of these circles equals the sum of the lengths of common tangents to two remaining pairs. Thus we obtain the same equality for the lengths of tangents from C to Ω_A, Ω_B and the length of a common tangent to these circles. Therefore one of common tangents to Ω_A, Ω_B passes through C.

   Second solution. Let ℓ be the common tangent to Ω_A, Ω_B, and Ω_C. Supposing that P lies on the arc AB let us prove that the reflection of ℓ about AB passes through C, i.e. C lies on the common tangent to Ω_A, Ω_B. It is sufficient to prove that the reflection C' of C about the midpoint M of AB lies on ℓ. Since $PC = PA + PB$, the distance from M to ℓ equal to $(PA + PB)/2$ is twice as little than the distance from C to ℓ. Therefore reflecting C about M we obtain a point lying on ℓ (fig. 8.7).

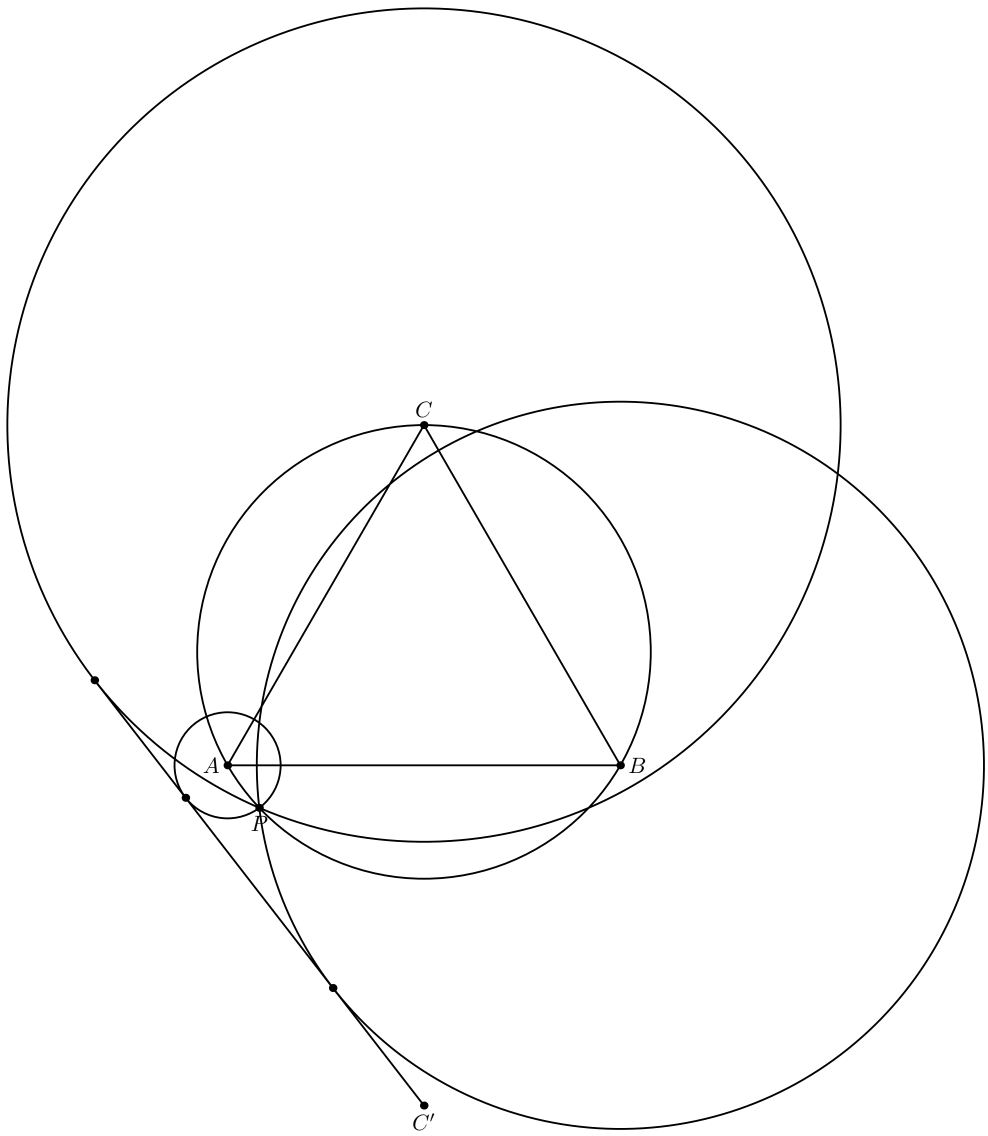

Fig. 8.7.

Remark. Any point P satisfying the assumption is a common point of the circumcircle and some excircle of the triangle. In this case three circles centered at A, B, C, and passing through P have a common tangent for an arbitrary triangle.

8. (A.Blinkov) Let $ABCDE$ be a paper pentagon with $AB = AE$, $\angle A = \angle B = \angle E = 90°$, $BC = 3$, $CD = 5$, $DE = 2$. Construct a perpendicular from A to CD using only a ruler and drawing not more than six lines. All lines have to be drawn inside the pentagon.

   Solution. A circle centered at A with radius AB touches the lines BC and DE. Since $CB + DE = CD$, this circle touches also the line CD, i.e. CA and DA are the bisectors of angles C and D respectively, and $\angle CAD = 45°$.

Let AK be the altitude of triangle ACD, and M, N be the common points of BE with AC and AD respectively. Then $BC{=}CK$, $DE = DK$, and $\angle CKM = \angle CBM = \angle DEN = \angle DKN = 45^\circ = \angle A$. Therefore DM and CN are the remaining altitudes of triangle ACD (fig. 8.8). From this we obtain the following construction

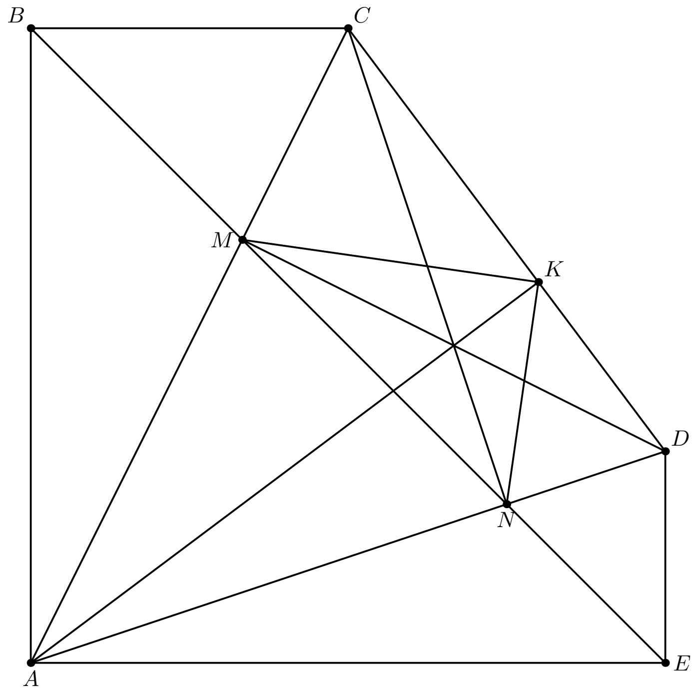

Fig. 8.8.

1–3. Draw the lines AC, AD, BE and mark the points M, N.

4–5. Draw the lines DM, CN and mark their common point H.

6. Draw the required line AH.

XXI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. First day. 9 grade.

July 31, 2025.

1. (Ya.Scherbatov) Altitudes AA_1, BB_1 of a triangle ABC meet at point H. Let A', B' be the reflections of A, B about BB_1, AA_1 respectively. Prove that the nine-points circles of triangles $A'B'C$ and $A'B'H$ are tangent.

   Solution. Let K, M_1, N_1, M_2, N_2 be the midpoints of $A'B'$, CB', CA', HB', HA' respectively. Then we have to prove that $\angle M_1KM_2 = \angle M_1N_1K + \angle M_2N_2K$. Since $\angle M_1KM_2 = \angle CA'H$, $\angle M_1N_1K = \angle KB'C$, and $\angle KN_2M_2 = \angle HB'K$, this is equivalent to the equality of angles $CA'H$ and $HB'C$. But $\angle HA'A = \angle A'AH = \angle HBB' = \angle BB'H = \pi/2 - \angle C$, which yields the required equality (fig. 9.1).

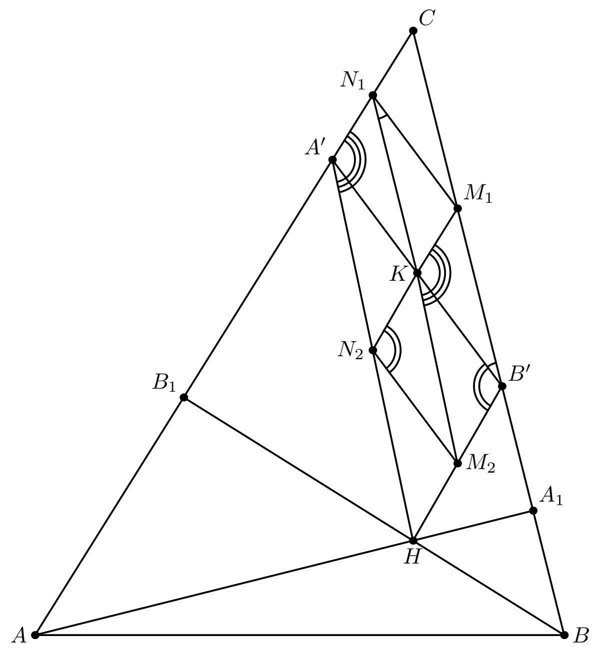

Fig. 9.1.

2. (F.Nilov) On the plane, several points are marked and colored into four colors so that any three points of different colors are not collinear, and any circle through three marked points of different colors contains exactly one marked point of the remaining color. Is it necessary that all marked points are concyclic?

   Answer. No, it is not.

Solution. Consider the complete quadrilateral formed by lines A_1B_1, A_1B_2, A_2B_1, A_2B_2. Let the lines A_1B_1 and A_2B_2 meet at point C_1 and the lines A_1B_2 and A_2B_1 meet at point C_2. Then the circles $A_1B_1C_2$, $A_1B_2C_1$, $A_2B_1C_1$, $A_2B_2C_2$ meet at the Miquel point D_2. Make an inversion centered at an arbitrary point D_1 not lying on constructed lines and circles, and color the maps of A_1, A_2 into the first color, the maps of B_1, B_2 into the second one, the maps of C_1, C_2 into the third color, D_1 and the map of D_2 into the fourth one. These eight points satisfy the assumption.

Remark. Another solution may be obtained from the addition of points on a circular cubic. Let A, B, C, D be the common points of such cubic with an arbitrary circle, and K_1, K_2, K_3 be three points of the cubic such that $2K_i = 0$. Then coloring A, $A + K_i$ into the first color, B, $B + K_i$ into the second one, C, $C + K_i$ into the third color, and D, $D + K_i$ into the fourth one we obtain 16 points satisfying the assumption.

3. (L.Emelyanov) A triangle ABC is given. A line m_1 meets BC, CA, AB at points A_1, B_1, C_1 respectively, and a line m_2 meets BC, CA, AB at points A_2, B_2, C_2, so that A_1 and A_2 are symmetric about the midpoint of BC, B_1 and B_2 are symmetric about the midpoint of CA, C_1 and C_2 are symmetric about the midpoint of AB. Prove that $m_1 \perp m_2$ if and only if m_1 and m_2 are two Simson lines of triangle ABC (for some two points of the circumcircle of ABC).

   First solution. The lines $A_1B_1C_1$ and $A_2B_2C_2$ generate a linear family of triangles $A_tB_tC_t$, where A_t, B_t, C_t divide the segments A_1A_2, B_1B_2, C_1C_2 in the same ratio. This family contains the medial triangle $A_0B_0C_0$ and two degenerate triangles $A_1B_1C_1$ and $A_2B_2C_2$.

   If two triangles of such family are orthologic, then two arbitrary triangles of the family are also orthologic. Hence $m_1 \perp m_2$ yields that $A_1B_1C_1$ and $A_0B_0C_0$ are orthologic, i.e. the perpendiculars from A_1, B_1, C_1 to the corresponding sidelines of ABC concur at some point P. Then P lies on the circumcircle of ABC and m_1 is its Simson line. Similarly m_2 is the Simson line of the opposite point on the circumcircle. Conversely, if m_1, m_2 are Simson lines, then $A_tB_tC_t$ are pedal triangles of a linearly moving point P_t. Since this family contains the medial triangle, the corresponding line passes through the circumcenter O, thus m_1 and m_2 are perpendicular.

   Second solution. Let the homothety centered at the centroid of ABC with coefficient $-1/2$ map m_2 to a line m_2'. Note that this homothety maps A_2,

B_2, C_2 to the midpoints of AA_1, BB_1, CC_1. Therefore m_2' is the Gauss line of the quadrilateral formed by the sidelines of ABC and m_1. It is known that the Gauss line is perpendicular to the Simson line of the Miquel point, i.e. it is parallel to m_1. This is possible only if m_1 coincides with the Simson line of the Miquel point. Since the Miquel point lies on the circumcircle of ABC, we obtain the required.

Remark. The Simson lines of any two opposite points satisfy the assumption (fig. 9.3).

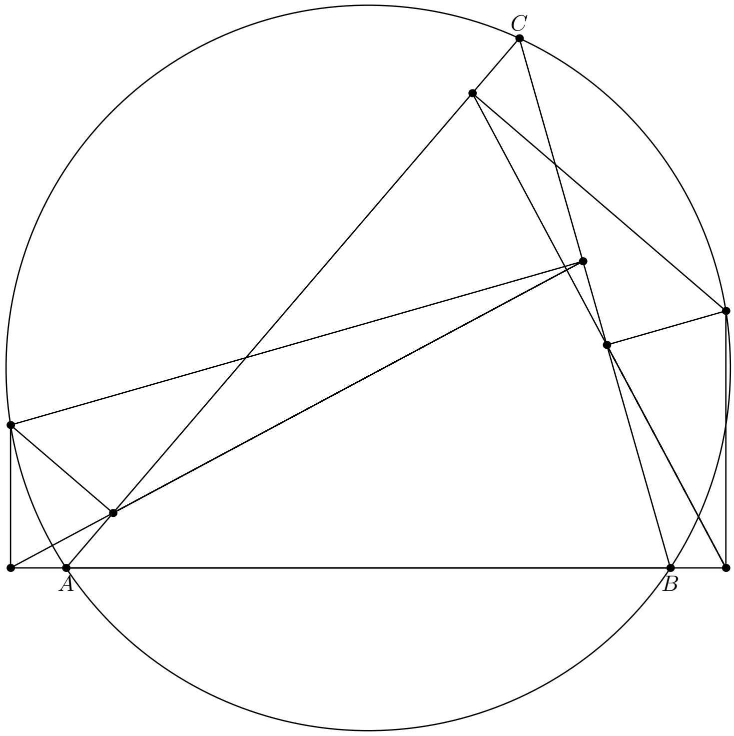

Fig. 9.3.

4. (E.Volokitin) Let $ABCD$ be a cyclic quadrilateral. Two lines passing through the orthocenter H of the triangle ABC and parallel to BD and CD meet AC and AB respectively at points E and F. Prove that the line EF bisects the segment DH.

   First solution. The midpoint of DH is the center of an equilateral hyperbola $ABCDH$. Let B', C' be the points of this hyperbola opposite to B, C respectively. Then $HB' \parallel BD$, $HC' \parallel CD$, and applying the Pascal theorem to the hexagon $ABB'HC'C$ we obtain the required.

   Second solution. Let O be the circumcenter of ABC. Note that $\angle EHF = \angle A$, hence there exists a point isogonally conjugated to H with respect to the quadrilateral $BFEC$, and this point coincides with O. Let X and C_1 be projections of H to BF and EF respectively. Denote by M and C' the projection of O to EF and the midpoint of CH respectively. Then M, C', X, C_1 lie on the nine-points circle of triangle ABC, because the pedal circles of O and H with respect to $BEFC$ coincide. From the Fuss lemma we obtain that $FH \parallel C'M$, thus the homothety centered at H with coefficient 2 maps M to D.

Third solution. Let H_b, H_c be the meeting points of altitudes of ABC with the circumcircle; Q_b, Q_c be the projections of D to AC, AB respectively; D_b, D_c be the meeting points of these perpendiculars with the circumcircle; P_b, P_c be their common points with the altitudes. Note that BD_bDH_b is an isosceles trapezoid, thus $\angle DBH_b = \angle EH_bH = \angle EHH_b$, i.e. $HE \parallel BD$, and E lies on H_bD_b. Similarly F lies on H_cD_c. Applying the Pascal theorem to the hexagon $H_bD_bDACH_c$ we obtain that EP_c passes through the common point of AD and H_bH_c (fig. 9.4). Similarly P_bF passes through this point. Then by the Desargues theorem the triangles FP_cQ_c and EP_bQ_b are perspective, but P_bP_c bisects DH as a diagonal of parallelogram HP_bDP_c, and Q_bQ_c bisects it as the Simson line of D.

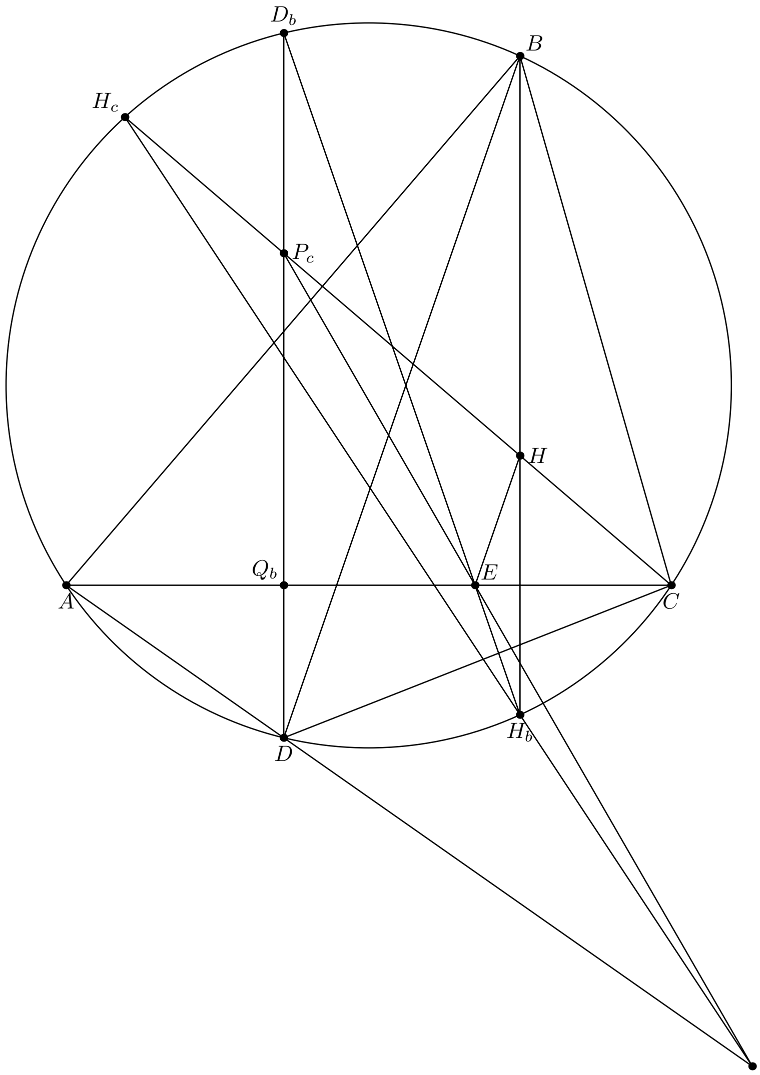

Fig. 9.4.

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Final round. Second day. 9 grade.

August 1, 2025.

5. (I.Mikhailov) Let BE and CF be altitudes of a triangle ABC. The internal bisectors of angles B and C meet at point I, and the external ones meet at point J. Prove that $IJ > EF$.

   Solution. By the trident theorem B and C lie on the circle with diameter IJ, therefore $BC \leq IJ$. On the other hand, from the similarity of triangles ABC and AEF we have $EF = BC \cos \angle A < BC$.

6. (ALEPH[1]/D.Brodsky) A triangle ABC is inscribed into a circle ω. The tangents to ω at points B and C meet at point S. The segments AS and BC meet at point P. The bisectors (the rays) of angles APC and SPC meet ω at points X and Y respectively. Prove that X, Y, and S are collinear.

   Solution. Let us prove a general assertion: if a line passing through S meets ω at points U, V, then the quadruple of lines PA, PB, PU, PV is harmonic.

   Consider a projective transformation fixing ω and mapping P to its center. It maps BC and AS to perpendicular diameters of the circle, and maps U, V to points symmetric about BC. The required assertion is evident.

7. (ALEPH/N.Shteinberg, A.Naumenja) A triangle ABC is given. Let D be an arbitrary point on the perpendicular bisector to BC, lying outside the triangle. The lines BD and AC meet at point C', and the lines CD and AB meet at point B'. A point M_a is the midpoint of BC, and M is the second common point of circles $(BB'D)$ and $(CC'D)$. Prove that the circumcenter of DMM_a lies on a fixed line.

   Solution. Let H_a be the common point of the A-Apollonius circle of ABC and the median AM_a. Let us prove that H_a, D, M_a, M are concyclic, this clearly yields the required assertion.

   Since M is the Miquel point of lines AB', CB', AC', BC', we obtain that $ABM \sim MDC$, therefore $\frac{MB}{AB} = \frac{MD}{DC} = \frac{MD}{DB} = \frac{MC}{AC}$, i.e. M lies on the Apollonius circle.

   Let A' be the reflection of A about BC, A' also lies on the Apollonius circle.

[1] https://aleph-problems.com/

Let us prove that A', M, D are collinear. When D moves along the perpendicular bisector to BC, M projectively moves along the circle, thus it is sufficient to prove this for three positions of D.

If D is infinite, then M coincides with A, and the assertion is correct.

If D is the circumcenter of $A'BC$, then M coincides with A', because the triangles $A'DC$ and $A'BA$ are similar. Also the tangent to the Apollonius circle at A' passes through D, because this circle is perpendicular to the circumcircle of $A'BC$.

If D lies on the circle $A'DC$, then $A'D$ is the bisector of triangle $BA'C$, and M is the foot of this bisector because the triangles MDC, MBA', and MBA are similar.

Finally note that $M_aD \parallel AA'$, thus by the inverse Fuss lemma H_a, M_a, M, D are concyclic (fig.9.7).

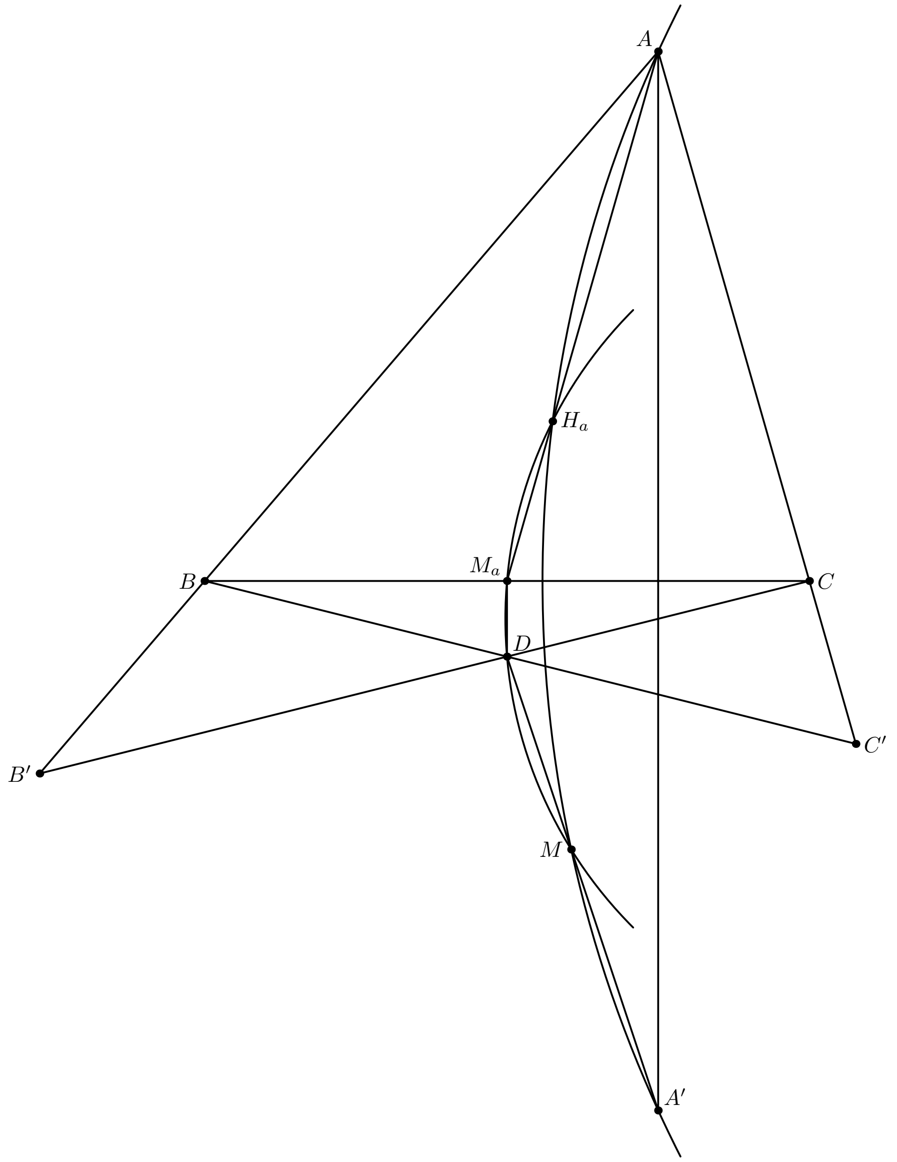

Fig. 9.7.

8. (S.Arutyunyan) Restore a bicentral quadrilateral $ABCD$ by the incenter I, the common point E of tangents to the circumcircle at points A, C, and the common point F of tangents to the circumcircle at points B, D.

   Solution. Let us prove two facts.

   Lemma 1. $EI \parallel BD$, $FI \parallel AC$.

   Proof. Suppose that $AB \geq AD$. Then

$$\angle AIC = \angle IAB + \angle ICB + \angle B = \frac{\pi}{2} + \angle B = \pi - \frac{\angle AEC}{2},$$

   therefore E is the circumcenter of AIC, and

$$\angle(IE, AC) = \angle EIC + \angle ICA = \frac{\pi}{2} - \angle IAC + \angle ICA =$$

$$= \frac{\pi}{2} - \frac{\angle BAC - \angle CAD}{2} + \frac{\angle BCA - \angle ACD}{2} = \frac{\smile BC + \smile AD}{2} = \angle(BD, AC).$$

Similarly $FI \parallel AC$.

Lemma 2. Let AC and BD meet EF at points K, L respectively. Then $\angle EIK = \angle FIL = \pi/2$.

Proof. Let O be the circumcenter of the quadrilateral, and OI meet EF at point P. Since EF is the polar of the common point of diagonals, lying on OI, we have $EF \perp OI$, and P lies on the circle with diameter EI, touching the circle AIC by lemma 1. Also A, C, P lie on the circle with diameter OE. Thus K is the radical center of circles ACE, IPE, AIC, and $\angle EIK = \pi/2$. Similarly $\angle FIL = \pi/2$.

From these lemmas we obtain the following construction.

1. Draw the lines through I perpendicular to EI, IF and mark the points K, L.

2. Draw the lines k, ℓ passing through K, L and parallel to IF, IE respectively.

3. Construct the orthocenter O of triangle IEF and the circle with diameter OE, it meets k at points A, C. Similarly construct the points B and D (fig. 9.8).

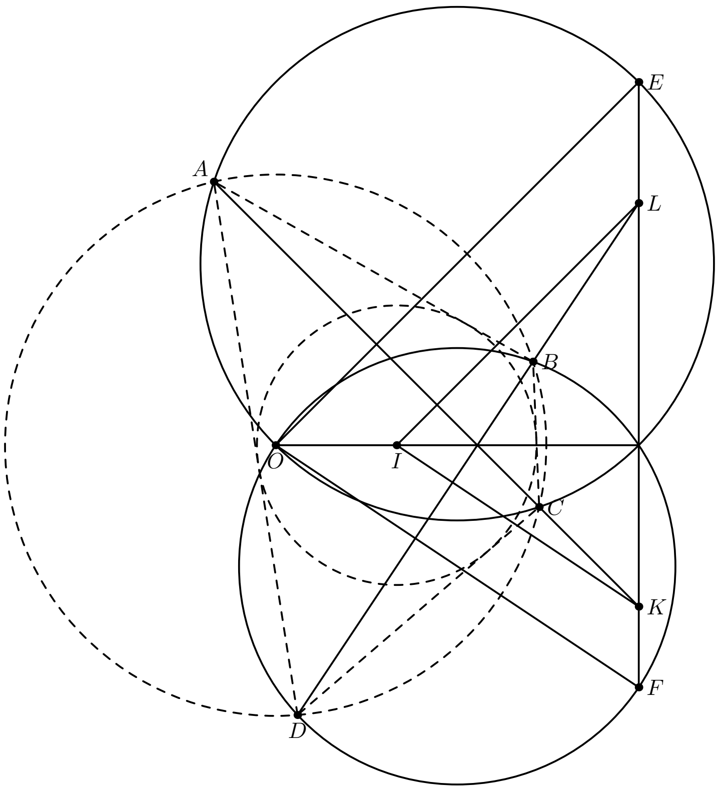

Fig. 9.8.

XXI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
Final round. First day. 10 grade.
July 31, 2025.

1. (I.Bogdanov) Two parallelograms $KL_1M_1N_1$ and $KL_2M_2N_2$ are inscribed into a convex quadrilateral $ABCD$ in such a way that K is the midpoint of AB, L_1, M_1, N_1 and L_2, M_2, N_2 lie on the sides BC, CD, DA respectively. Can the area of one parallelogram be less than a half of the area of the quadrilateral, and the area of the second one be greater than the half of the area of the quadrilateral?

   Answer. No, they cannot.

   Solution. Suppose that $\angle A + \angle B < \pi$. Fix an arbitrary parallelogram $KLMN$ such that N lies on the segment AP, L lies on the segment BP, and M lies inside the triangle ABP, where P is the common point of the lines AD and BC. Let an arbitrary line passing through M meet the segments BP, AP at a points C', D' respectively (fig. 10.1). It is known that the area of triangle $PC'D'$ is minimal, when M bisects $C'D'$. In this case $KLMN$ is the Varignon parallelogram of the quadrilateral $ABC'D'$ and its area equals a half of the area of this quadrilateral. Thus the areas of both parallelograms $KL_1M_1N_1$ and $KL_2M_2N_2$ are not less than the half of the area of $ABCD$.

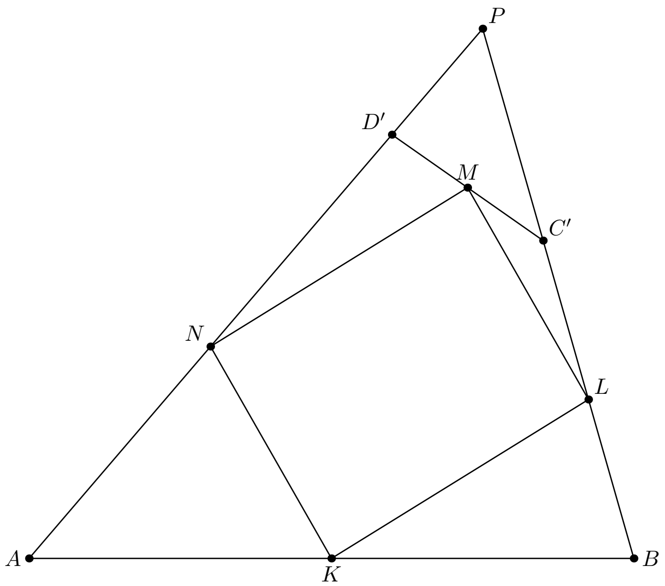

Fig. 10.1.

Similarly we obtain that if $\angle A+\angle B > \pi$, then the areas of both parallelograms are not greater than the half of the area of the quadrilateral. Finally, if

$AD \parallel BC$, then the areas of both parallelograms are equal to the half of the area of the quadrilateral.

2. (E.Volokitin) Let I be the incenter of a scalene triangle ABC; and P, Q be two isogonal points such that $AP \parallel IQ \parallel BC$. Prove that $AP = |AB - AC|$.

   Solution. For any point P' such that $AP' \parallel BC$ the sum of oriented areas of triangles $P'AB$ and $P'CA$ equals zero, and the ratio of these areas and the area of triangle $P'BC$ equals $\pm AP'/BC$. Take a point P' such that the ratio of oriented areas is $S_{P'BC} : S_{P'CA} : S_{P'AB} = a : (b-c) : (c-b)$. Then for the isogonal point Q' we have $S_{Q'BC} : S_{Q'CA} : S_{Q'AB} = a : b^2/(b-c) : c^2(c-b)$, therefore $S_{Q'BC} : S_{ABC} = a : (a+b+c)$, i.e. $Q'I \parallel BC$. Hence P', Q' coincide with P, Q.

   Remark. Here is an idea of another solution.

   Let a, b, c, p be the sidelengths and the semiperimeter of ABC. Denote by X the second common point of IQ and the circumcircle of BIC. Then $BIXC$ is an isosceles trapezoid. Therefore the projection of IX to BC equal $|(p-b)-(p-c)| = |b-c|$. Hence it is sufficient to prove that $IX = AP$. Let X' be isogonally conjugated to X with respect to ABC. Then X and X' are symmetric about AI, thus we have to prove that P and X' are symmetric about the perpendicular bisector to AI. Let $\mathcal{C}$ be the isogonal map of IQ with respect to ABC. Then A, B, C, I, X', P lie on the hyperbola $\mathcal{C}$, and the lines AQ and IQ touch it. Since $IQ = AQ$, we obtain that the perpendicular bisector to AI is an axis of $\mathcal{C}$. Then the reflection about it maps P to X'.

3. (E.Alkin, A.Skopenkov) Do there exist six point $A_1, \dots, A_6$ in general position in the space, such that the triangles $A_1A_2A_3$ and $A_4A_5A_6$ are linked, and two triangles corresponding to any other dividing of these points into two triplets are not linked. *Two triangles on the space are linked if the outline of one triangle meets the inside of the second one at a unique point.*

   Answer. Yes, they do.

   Solution. Take two congruent regular pyramids with common base $A_1A_2A_3$ and vertices A_4, A_5 symmetric about the base. Take point A_6 on the plane passing through A_1A_4 and the altitude of triangle $A_5A_2A_3$, inside the angle vertical to the angle containing A_1, A_4, A_5 (fig. 10.3). Then all segments $A_6A_i, i = 1, \dots, 5$, lie outside the bipyramid $A_1A_2A_3A_4A_5$, and the points $A_1, \dots, A_6$ satisfy the assumption.

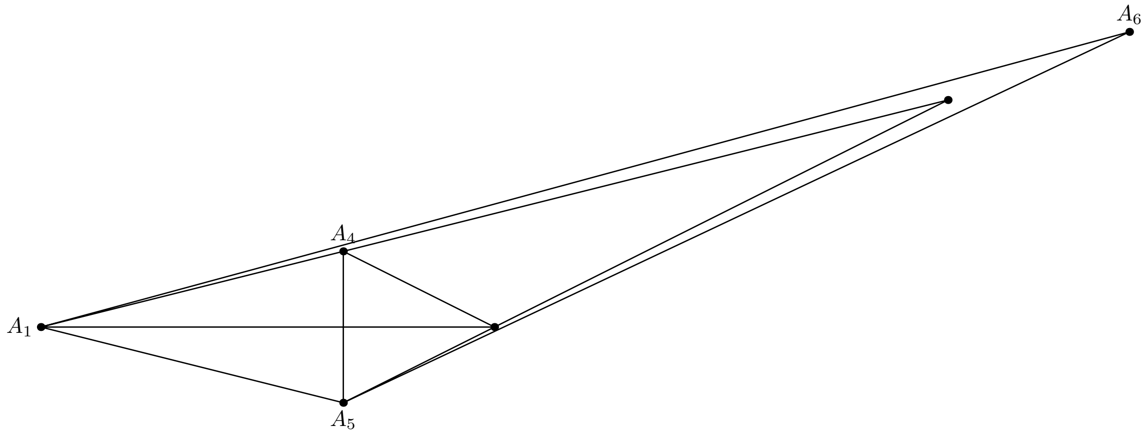

Fig. 10.3.

4. (S.Kuznetsov) Let M, H, L be the centroid, the orthocenter, and the Lemoine point respectively of a triangle ABC. A point S is such that the circles SLH, SML touch MH, and L' is the reflection of L about the circumcircle of the triangle. Prove that $SL' \parallel MH$.

   First solution. Use the following property of an equilateral hyperbola.

   Let points A, B lie on an equilateral hyperbola centered at O. The tangents to the hyperbola at A, B meet at point S. Then O is the Humpty point of triangle SAB corresponding to S.

   In fact, construct a rectangle $AUBV$ with the sides parallel to the asymptotes of the hyperbola. The points U, V, O, S are collinear and form a harmonic quadruple, therefore O and S are inverse about the circle with diameter AB.

   Apply this property to the Kiepert hyperbola $ABCMH$. The tangents to it at M, H meet at L, therefore S is the center of the hyperbola, i.e. the midpoint of segment T_1T_2 between two Torricelli points. On the other hand L' is the midpoint of segment $T_1'T_2'$ between two Apollonius points. But $T_1T_1' \parallel T_2T_2' \parallel MH$ (fig. 10.4).

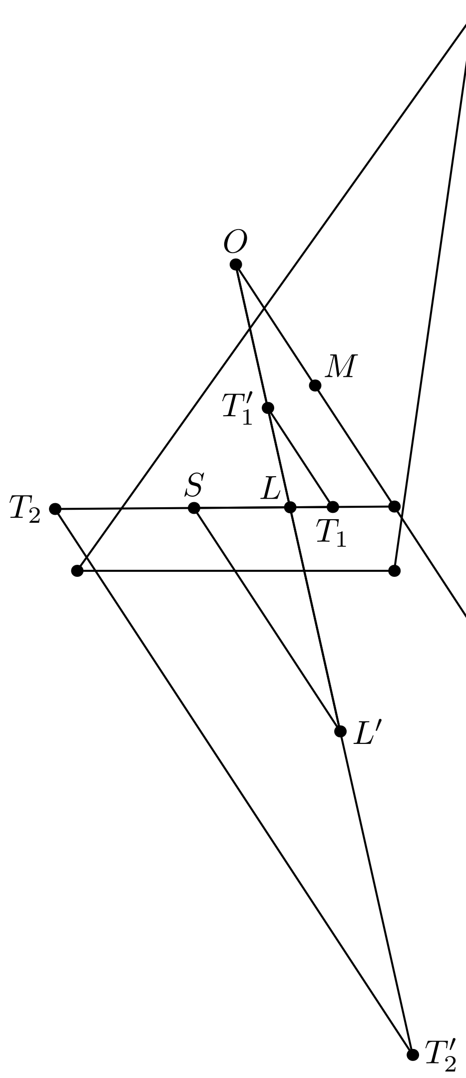

Fig. 10.4.

Second solution. Let Γ be an ellipse with foci O, H inscribed into ABC. Rotate ABC by Poncelet between its circumcircle and Γ. We have that H is the orthocenter of all triangles ABC as the isogonal image of the circumcenter O, hence the centroid M and the nine-points circle of ABC are also fixed. The point S lies on the nine-points circle, because it is the center of the Kiepert hyperbola, therefore its inversion image — the point L moves along a circle Δ with a center lying on MH, and O is the reflection of the midpoint of MH about this circle. The point L' as the inversion image of L about the circumcircle of ABC also moves along some circle, and the angle velocities of L' and S are equal. It is easy to see that the radius of this circle equals the radius of the nine-points circle. Therefore $SL' \parallel MH$.

Remark. The points S and L' are symmetric about the circumcircle and the point M.

XXI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN

Final round. Second day. 10 grade.

August 1, 2025.

5. (K.Belsky) Let P be a random point inside a regular triangle ABC. Find the probability that there exists an acute-angled triangle with the sidelengths AP, BP, CP.

Answer. $4 - \frac{2\pi}{\sqrt{3}}$.

Solution. Consider a rotation by $\pi/3$ about C, mapping A to B, and P to a point Q. We have $BQ = AP$, $PQ = CP$ (because CPQ is a regular triangle), therefore the sidelengths of triangle PBQ are equal to the lengths of segments AP, BP, CP. Also $\angle PQB = \angle CQB - \pi/3 = \angle CPA - \pi/3$ (fig.10.5.1). Similarly two remaining angles equal $\angle APB - \pi/3$ and $\angle BPC - \pi/3$. Thus this triangle is acute-angled if and only if all sides of ABC are seen from P at angles less than $5\pi/6$.

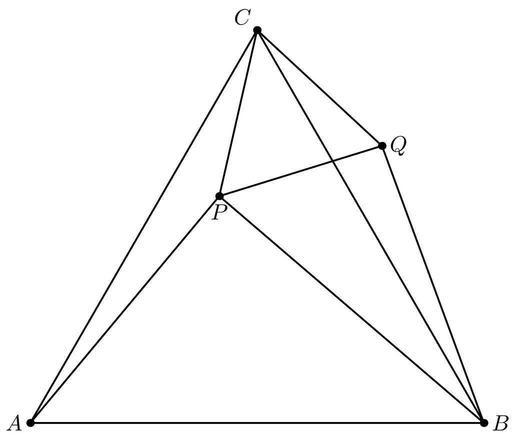

Fig. 10.5.1.

Let A', B', C' be the reflections of A, B, C about the opposite sidelines of ABC. Then AB is seen at the angle $5\pi/6$ from the points of arc centered at C' with radius $C'A = AB$. This arc and two arcs constructed similarly for the remaining sides are pairwise tangent at the vertices of ABC (fig.10.5.2). Hence the area of a curvilinear triangle limited by these arcs equals the difference of the area of $A'B'C'$ and the areas of three sectors with radius AB and angle $\pi/3$, i.e. $\sqrt{3} - \pi/2$ (if $AB = 1$). Dividing by the area of ABC we obtain the required probability.

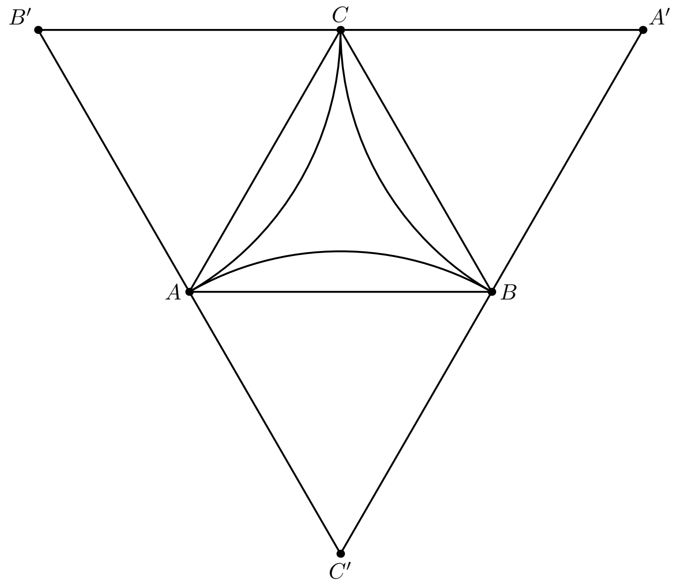

Fig. 10.5.2.

6. (Ya.Scherbatov) Circles Ω and ω_a are the circumcircle and the $A-$excircle of some triangle ABC. Let I_b, I_c be the centers of two remaining excircles, and A_b, A_c be the touching points of the extensions of AB, AC with ω_a. Prove that the meeting point of the lines A_bI_b and A_cI_c does not depend on the triangle ABC.

   Solution. Let Ω and ω_a meet at points P and Q, and the lines PQ and AB meet at point X. Since PQ is the radical axis of Ω and ω_a, and AB is the radical axis of Ω and the circle I_bABI_a, we obtain that X lies on the radical axis of ω_a and the circle with diameter I_bI_a, i.e. on the polar of I_b with respect to ω_a. Thus the polars with respect to ω_a of A_b and I_b meet on PQ, therefore I_bA_b passes through the common point Y of tangents to ω_a at P, Q. Similarly A_cI_c passes through Y, and Y clearly does not depend on ABC (fig.10.6).

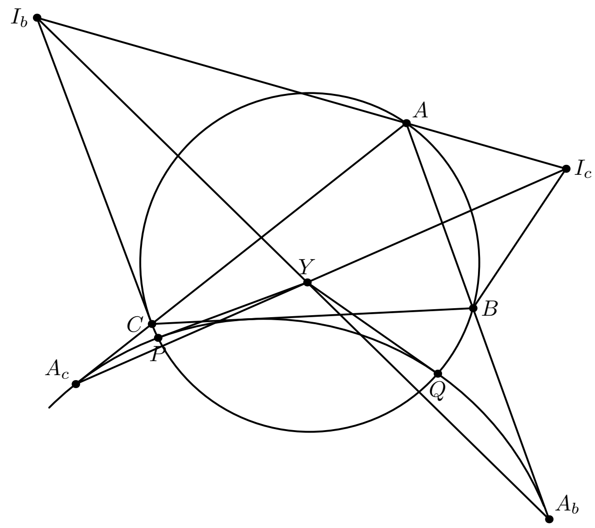

Fig. 10.6.

7. (M.Vekshin) Let $ABCD$ be a cyclic quadrilateral. An arbitrary conic passes through A, B, C, D. Consider four lines which are the isogonal maps of this conic with respect to the triangles ABC, ABD, BCD, ACD. Prove that the quadrilateral formed by these lines is circumscribed.

   First solution. An equation of an arbitrary conic passing through A, B, C, D is $tF_1(x,y)+(1-t)F_2(x,y)=0$, where $F_1(x,y)=0$, $F_2(x,y)=0$ are the equations of two parabolas passing through A, B, C, D. The isogonal maps of all these conics with respect to ABC are parallel lines because the isogonal map of D is infinite. Clearly, when t changes uniformly, the corresponding line also moves uniformly. But the isogonal maps of circumparabolas touch the circumcircle of $ABCD$. Therefore the maps of any conic touch the same circle concentric with the circumcircle of $ABCD$.

   Second solution. Denote by X and Y two infinite points of the given conic. Let X_a and Y_a be isogonally conjugated to X and Y with respect to the triangle BCD. Note that $\overset{\frown}{X_aY_a}=2\angle(X,Y)$, i,e. the length of this arc is the same for all four triangles. Since the isogonal maps of a conic cut off equal arcs, the distances from the circumcenter to these lines are equal.

   Remark. If a conic is an ellipse, then X, Y, X_a, Y_a are imaginary points, but the equality $\overset{\frown}{X_aY_a}=2\angle(X,Y)$ is correct.

   Third solution. Prove that the isogonal images of any conic about the triangles ABC and ABD are symmetric about the perpendicular bisector to AB, this and similar assertions yield, that the distances from the circumcenter to all lines are equal. The isogonal images of the conics about each triangle form a pencil of parallel lines. It is easy to see that the lines of both pencils form equal angles with AB, hence it is sufficient to prove that the corresponding lines meet at the perpendicular bisector. It is clear that the correspondence between the pencils is projective, and the infinite line corresponds to itself. By the Sollertinsky lemma it is sufficient to find two conics, such that their images meet on AB. This are two degenerated conics: $AC \cup BD$ and $AD \cup BC$.

8. (K.Belsky) Let ABC be an acute-angled triangle. A line ℓ meets the sides AB, AC and the sideline BC at points C_1, B_1, A_1 respectively. A circle ω_a touches BC at A_1 and touches the minor arc BC of the circumcircle of ABC. Circles ω_b, ω_c are defined similarly. Prove that these three circles have a common tangent.

   Solution. Let A_2, B_2, C_2 be the touching points of the circumcircle of ABC with ω_a, ω_b, ω_c respectively. Then from the Archimedes lemma and the property of a bisector we obtain that $\frac{BC_1}{AC_1} = \frac{BC_2}{AC_2}$. By the Ceva theorem and the Menelaos theorem for ℓ the lines AA_2, BB_2, CC_2 concur. Take an inversion centered at A. We obtain the following problem.

   The circle ω_b touches the lines AC, BC at points B_1, B_2, and the circle ω_c touches AB, BC at points C_1, C_2 respectively. Let A_2 be a point on BC such that $A_2C \cdot A_2C_2 = A_2B \cdot A_2B_2$. Let us prove that there exists a circle touching ω_a, ω_b, ω_c and passing through A.

   Construct a circle ω passing through A and touching internally ω_b and ω_c. Let us prove that ω and ω_a are tangent. Let ω meet BC at points X and Y. Let I_1 and I_2 be the incenters of triangles AXY and ABC. By the Sawayama lemma B_1B_2 and C_1C_2 meet at point I_1. Then the triangles BI_2C and $B_2I_1C_2$ re homothetic. This yields that A_2 lies on I_1I_2. Applying the inverse Sawayama lemma to the tangent from A to ω_a and the line BC we obtain the required (fig. 10.8).

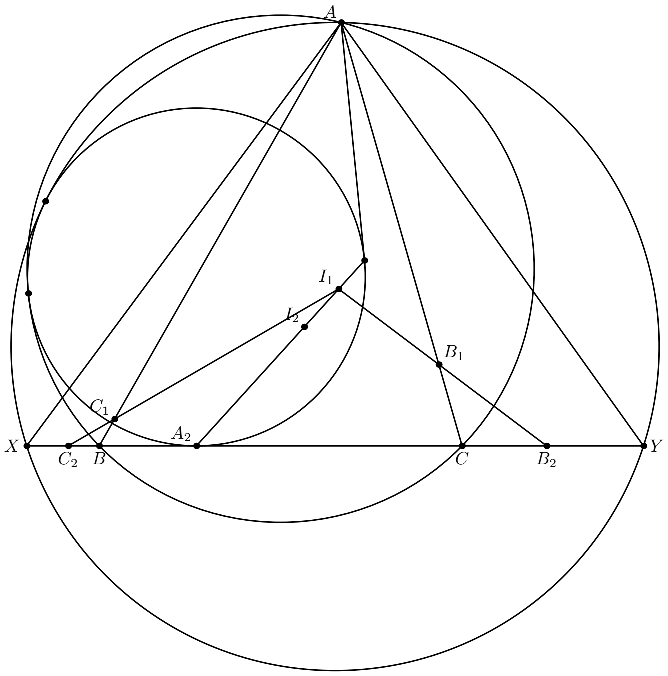

Fig. 10.8.