XXI GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round. Solutions

1. (8) (D.Shvetsov) Let I be the incenter of a triangle ABC, D be an arbitrary point of segment AC, and A_1, A_2 be the common points of the perpendicular from D to the bisector CI with BC and AI respectively. Define similarly the points C_1, C_2. Prove that B, A_1, A_2, I, C_1, C_2 are concyclic.

   Solution. Consider the configuration on fig. 1, for other cases the reasoning is similar. Since $DC_2 \perp AI$, we obtain that $\angle C_1C_2I = \angle DC_2I = \angle AIC - 90^\circ = \angle ABI = \angle C_1BI$, i.e. B, I, C_1, C_2 are concyclic. Similarly B, I, A_1, A_2 are concyclic. Also C_1 and A_1 are the reflections of D about AI, CI respectively, hence $\angle BC_1I = \angle IDC = \angle IA_1C$, and B, I, A_1, C_1 are concyclic. Thus all six points are concyclic.

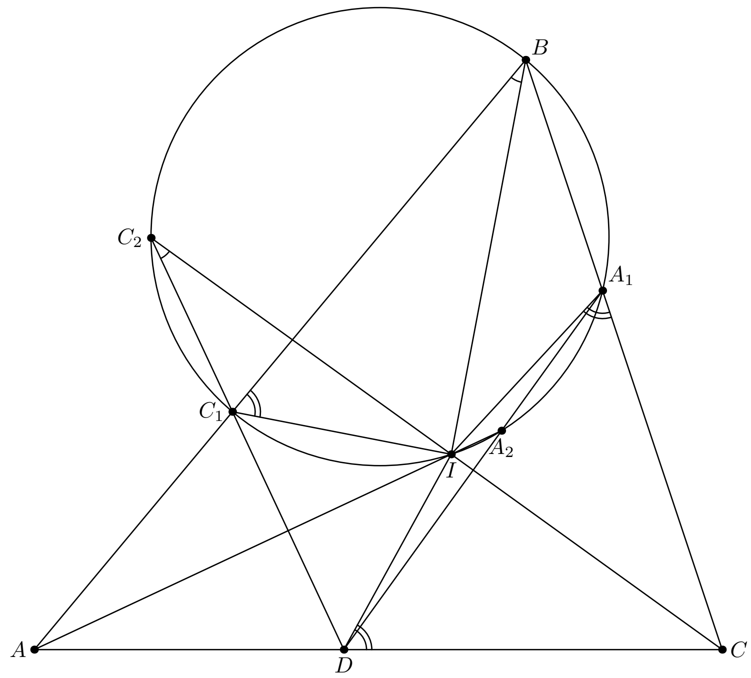

Fig. 1.

2. (8) (A.Kuznetsov) Four points on the plane are not concyclic, and any three of them are not collinear. Prove that there exists a point Z such that the reflection of each of these four points about Z lies on the circle passing through three remaining points.

   Solution. Let A, B, C, D be the given points; K, L, M, N, P, Q be the midpoints of segments AB, BC, CA, BD, CD, AD. The reflection of D about Z lies on the circle ABC if and only if Z lies on the circle NPQ. Hence we have to prove that the circles NPQ, KMQ, KLN, and LMP have a common point. Let Z be the second common point of circles KLN and NPQ (fig.2). Then $\angle LZN = \angle LKN = \angle CAD$ (we have the last equality because $KL \parallel AC$ and $KN \parallel AD$ as the medial lines of triangles ABC, ABD respectively). Similarly $\angle NZP = \angle BAC$, thus $\angle LZP = \angle BAC = \angle LMP$. Therefore Z lies on the circle LMP. Similarly we obtain that Z lies on the circle KMQ.

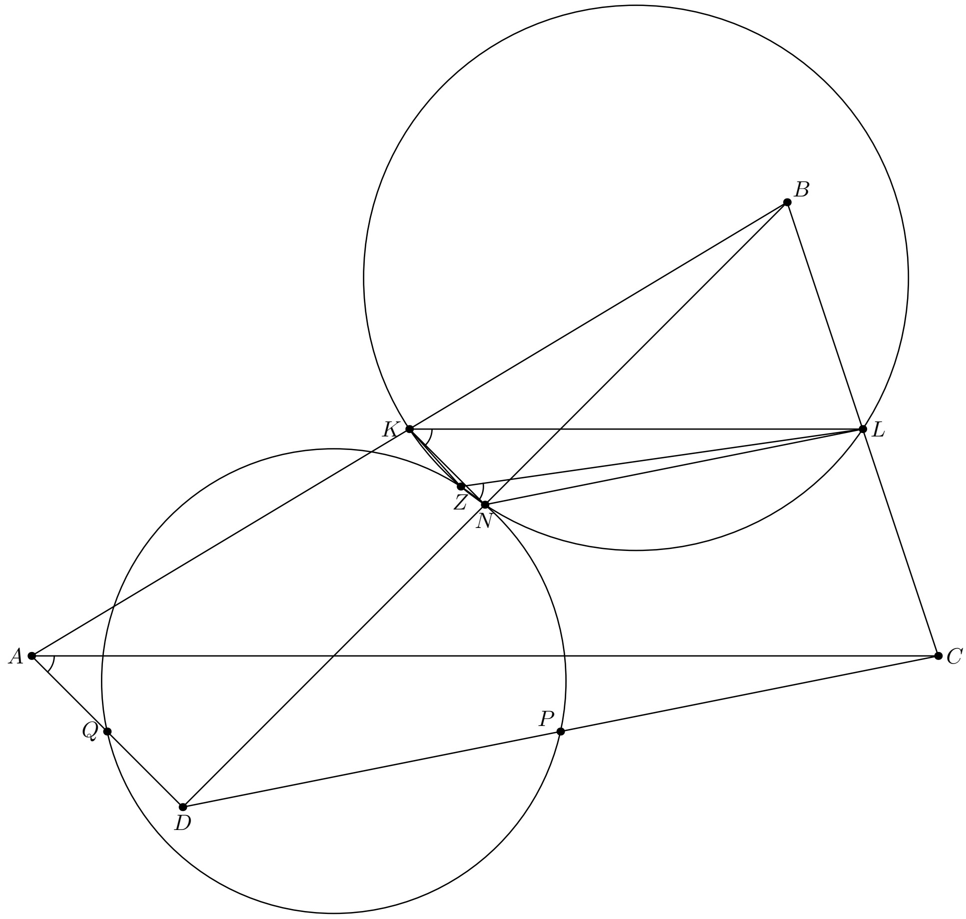

Fig. 2.

Remark. The assertion of problem may also be obtained from the fact that the nine-points circles or triangles ABC, BCD, CDA, DAB have a common point. The point Z is the reflection of this point about the centroid of A, B, C, D.

3. (8) (K.Belsky) An excircle centered at I_A touches the side BC of a triangle ABC at point D. Prove that the pedal circles of D with respect to the triangles ABI_A and ACI_A are congruent.

   Solution. Let P_1, P_2, Q_1, Q_2, R be the projections of D to AC, AB, I_AC, I_AB, AI_A respectively. Since P_1, P_2, R lie on the circle with diameter AD, and AR bisects the angle P_1AP_2, we obtain that $P_1R = P_2R$. Also since the quadrilaterals CP_1DQ_1 and I_AQ_1DR are cyclic, we obtain that $\angle P_1Q_1R = \angle P_1Q_1D + \angle DQ_1R = \angle P_1CD + \angle DI_AR = (\angle B + \angle C)/2$ (fig. 3). Similarly $\angle P_2Q_2R = (\angle B + \angle C)/2$. Since the equals chords of circles P_1Q_1R and P_2Q_2R correspond to equal angles, These circles are congruent.

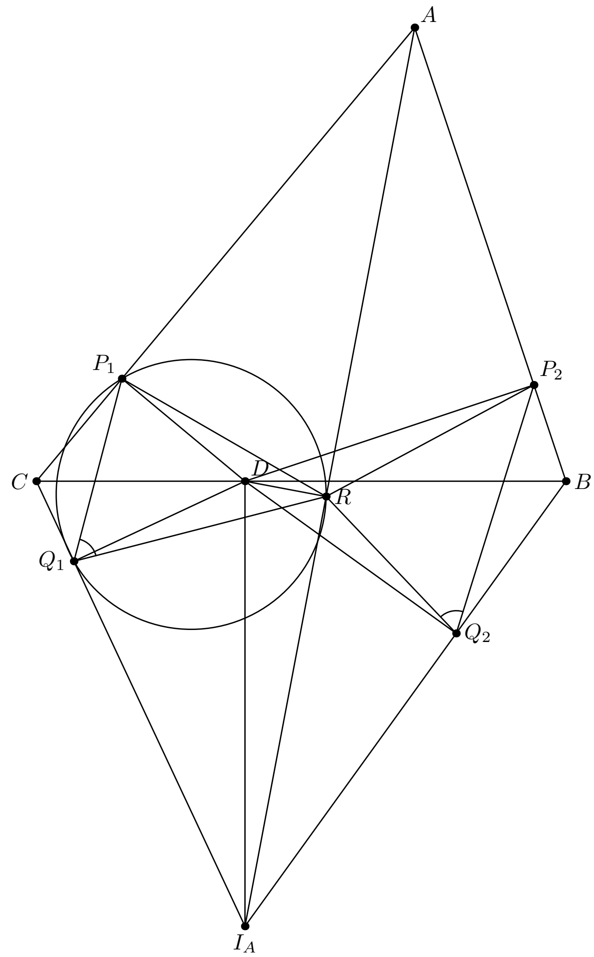

Fig. 3.

Remark. We can also to prove that the circles P_1Q_1R and P_2Q_2R are tangent at R.

4. (8) (Y.Shcherbatov) Let AL be the bisector of a triangle ABC, X be an arbitrary point on the external bisector of angle A, the lines BX, CX meet the perpendicular bisector to AL at points P, Q respectively. Prove that A, X, P, Q are concyclic.

   First solution. Let PQ meet AB at point Y. Then in the isosceles triangle AYL $\angle ALY = \angle LAY = \angle LAC$, therefore $LY \parallel AC$ and $XP : PB = AY : YB = YL : YB = AC : AB = CL : LB$. Hence $LP \parallel CX$. Similarly $LQ \parallel BX$, i.e. $LPXQ$ is a parallelogram (fig. 4). Thus $AQ = QL = XP$, i.e. the trapezoid $AXPQ$ is isosceles and cyclic.

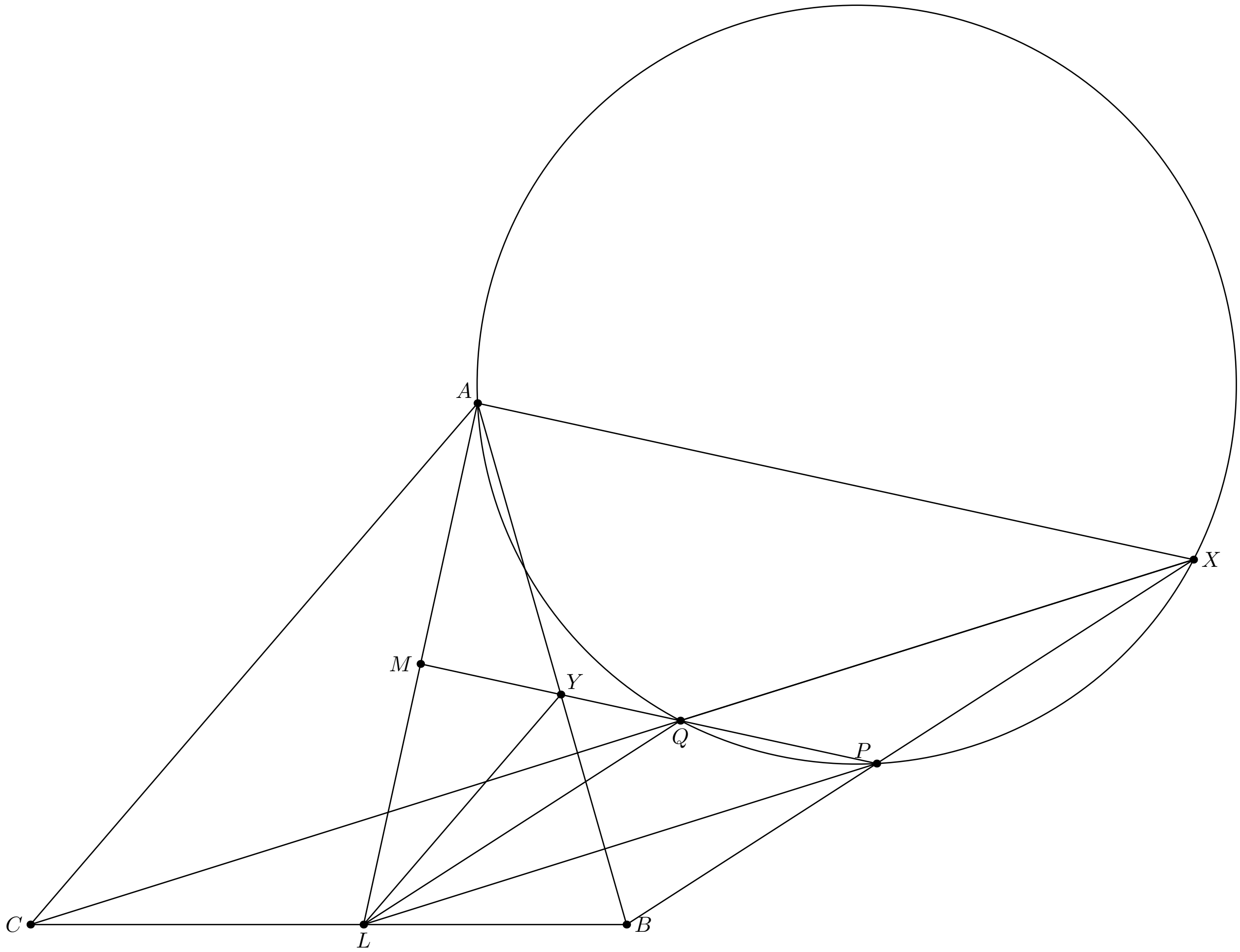

Fig. 4.

Second solution. Let M be the midpoint of AL; the perpendicular ℓ from L to AL meet BX and CX at points U and V respectively. If X moves uniformely along the bisector, then U, V also move uniformely along ℓ. If X coincides with A or the foot of the external bisector, then L bisects the segment UV. Thus this is correct for any point X. Thus since the medial lines of trapezoids $AXUL$ and $AXLV$ lie on PQ, we have $MP = (AX + LU)/2$, $MQ = (AX - LY)/2$. Hence $MP + MQ = AX$, which also yields that the trapezoid $AXPQ$ is isosceles.

Third solution. Let D and E be the common points of PQ with AB and AC respectively. Then $ADLE$ is a rhombus, and $DL \parallel AC$. From $BP : BX = BD : BA = BL : BC$ we have $PL \parallel XC$, $\angle LPQ = \angle QXA$, $\angle APQ = \angle LPM = \angle AXQ$, i.e. A, X, P, Q are concyclic.

5. (8) (D.Shvetsov) Let M be the midpoint of the cathetus AC of a right-angled triangle ABC ($\angle C = 90^\circ$). The perpendicular from M to the bisector of angle ABC meets AB at point N. Prove that the circumcircle of triangle ANM touches the bisector of angle ABC.

   Solution. Let P be the projection of M to the bisector of angle B, and O be the center of circle AMN. Since P and C lie on the circle with diameter BM, we have $\angle NMA = \angle PBC = \angle ABP$. Then $\angle AON = \angle ABC$ and $\angle MAO = 90^\circ - \angle ABC/2 - \angle BAC =$

$\angle NMA$, i.e. $AO \perp BP$. The angle between the tangent to the circle at point Q opposite to A and AB equals $90^\circ - \angle QAN = \angle QBA$, therefore this tangent coincide with BP (fig. 5).

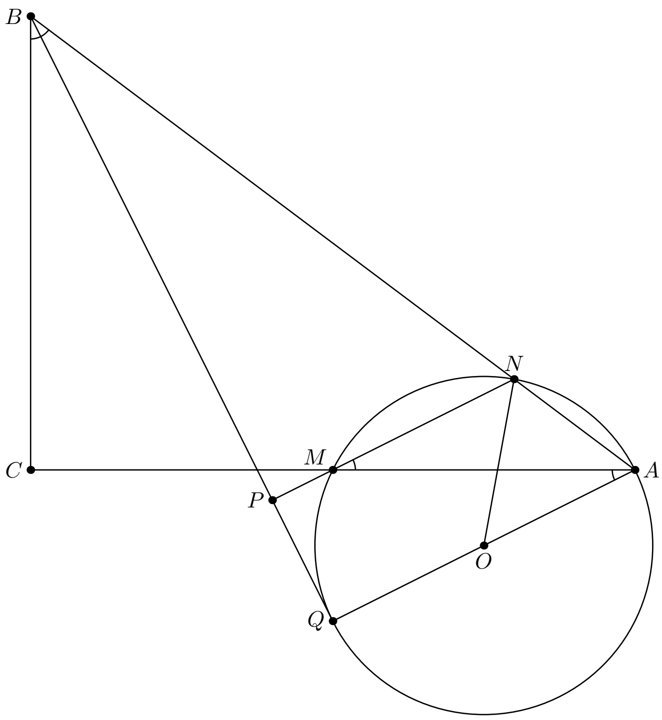

Fig. 5.

6. (8–9) (L.Emelyanov) One bisector of a given triangle is parallel to one sideline of its Nagel triangle. Prove that one of two remaining bisectors is parallel to another sideline of the Nagel triangle.

First solution. Let the sides BC, CA, AB of triangle ABC touch the corresponding excircles at points A', B', C' and the line $C'A'$ be parallel to the bisector AA_1 of angle A. Since $BC' = p - a$, $BA' = p - c$, $BA_1 = ac/(b + c)$, this is equivalent to the equality $a : (b + c) = (p - c) : (p - a)$, which can be transformed to $(p - a)(p - b) = p(p - c)$. We obtain the same equality if $B'C'$ is parallel to the bisector of angle B.

Second solution. Let 2α, 2β, 2γ be the values of the angles A, B, C; I, I' and I'' be the centers of the incircle A-excircle, and B-excircle respectively; $C'A' \parallel CII'$. It is known that $I'C' \perp AB$, $I'C \perp CI$, hence $CA' = AC' = r_a \operatorname{tg} \angle CI'A' = r_a \operatorname{tg} \gamma$. The distance from A' to AI equals $AC' \sin\gamma$. Since $\angle BI'A' = \beta$, $\angle BI'A = 180^\circ{}^{\smile}\alpha^{\smile}(90^\circ + \beta) = \gamma$, we have $\angle AI'A' = \gamma - \beta$. The distance from C' to AI' equals $I'C' \sin \angle AI'C' = r_a \sin(\gamma - \beta) = AC' \sin\alpha = r_a \operatorname{tg}\gamma \cos(\gamma + \beta)$. Thus $sin(\gamma - \beta)\cos\gamma = \cos(\gamma + \beta)\sin\gamma$. Therefore $\sin^2\gamma \sin\beta = \cos^2\gamma \sin\beta$ and $\gamma = 45^\circ$, $\angle C = 90^\circ$, $\angle B'CI'' = \angle CI''B' = 45^\circ$, which yields $BA' = AB' = B'I''$, $\angle BI''B' = 45^\circ - \angle CI''B = \beta$. This $A'BI''B'$ is an isosceles trapezoud, and $C'B' \parallel BI$.

Remark. The assumption of the problem is correct for any right-angled triangle (fig. 6).

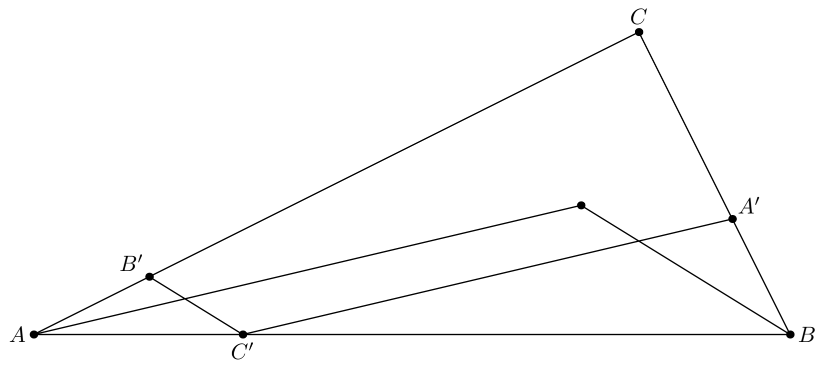

Fig. 6.

7. (8–9) (Y.Shcherbatov) Let I, I_a be the incenter and the A-excenter of a triangle ABC; E, F be the touching points of the incircle with AC, AB respectively; G be the common point of BE and CF. The perpendicular to BC from G meets AI at point J. Prove that E, F, J, I_a are concyclic.

   Solution. Let J' be the common point of AI with the circle I_aEF, the sideline BC meet I_aF, I_aE at points X, Y and touch the incircle at point D. The triangles $J'EF$ and IXY are orthologic, because I_a is their orthology center. Also $-1 = (A, BC \cap AI, I, I_a) = (A, B, XI \cap AB, F)$, therefore XI passes through the common point of ED and AB. Then $IX \perp CF$ (fig. 7). Similarly $IY \perp BE$. Thus $J'G \perp XY$, which yields the required assertion.

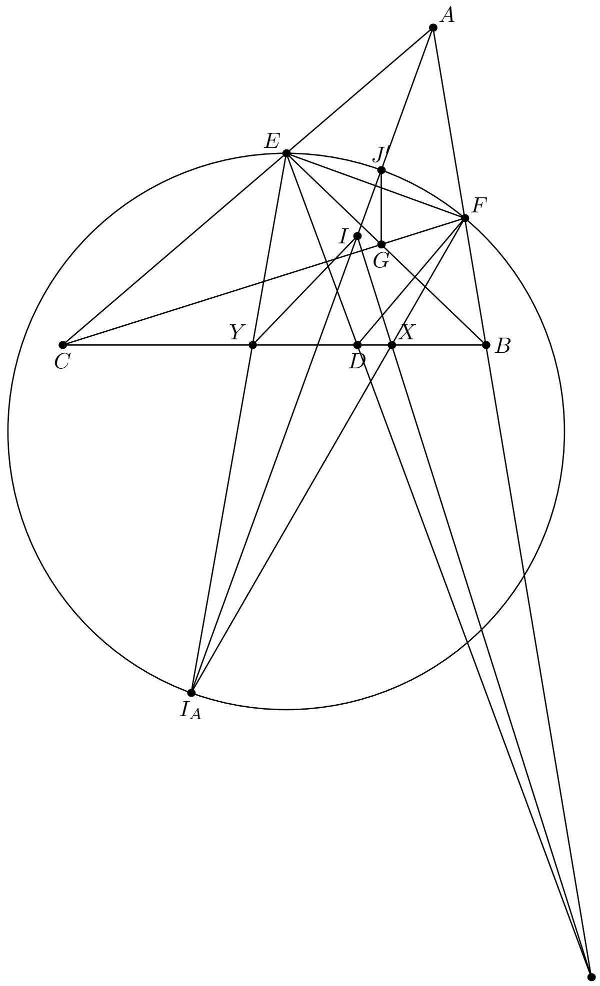

Fig. 7.

8. (8–9) (В.Конышев) (V.Konyshev) The diagonals of a cyclic quadrilateral $ABCD$ meet at point P. Points K and L lie on AC, BD respectively in such a way that $CK = AP$ and $DL = BP$. Prove that the line joining the common points of circles ALC and BKD passes through the mass-center of $ABCD$.

   Solution. Let M, N be the midpoints of AC and BD respectively, Q be the second common point of AC with the circle BKD (fig. 8). Then the degrees of M with respect to the circles ALC and BKD are equal to $AC^2/4$ and $MK \cdot MQ$ respectively. Also $MK = MP = PK/2 = (PA-PC)/2$, $MQ = PB{\cdot}PD/PK = PA{\cdot}PC/(PA-PC)$. From this we obtain that the difference of degrees equals $AP \cdot PC/4$. Similarly the difference of degrees of N with respect to these circles equals $-PB \cdot PD/4 = -PA \cdot PC/4$. Since the difference of degrees is a linear function, the midpoint of MN lies on the radical axis.

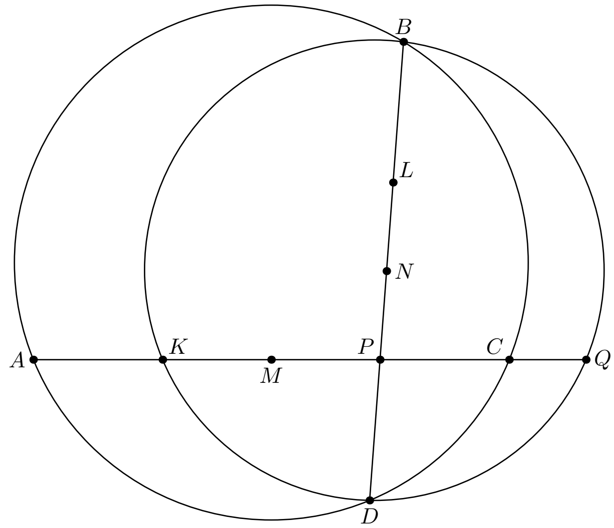

Fig. 8.

9. (8–9) (A.Mardanov, K.Mardanova) The line ℓ passing through the orthocenter H of a triangle ABC ($BC > AB$) and parallel to AC meets AB and BC at points D and E respectively. The line passing through the circumcenter of the triangle and parallel to the median BM meets ℓ at point F. Prove that the length of segment HF is three times greater than the difference of FE and DH.

   Solution. Let BM meet DE at point N (fig. 9). Then $NF = HN/2$ and $DH + HN = NE = NF + EF$. Therefore $EF - DH = NH/2 = HF/3$.

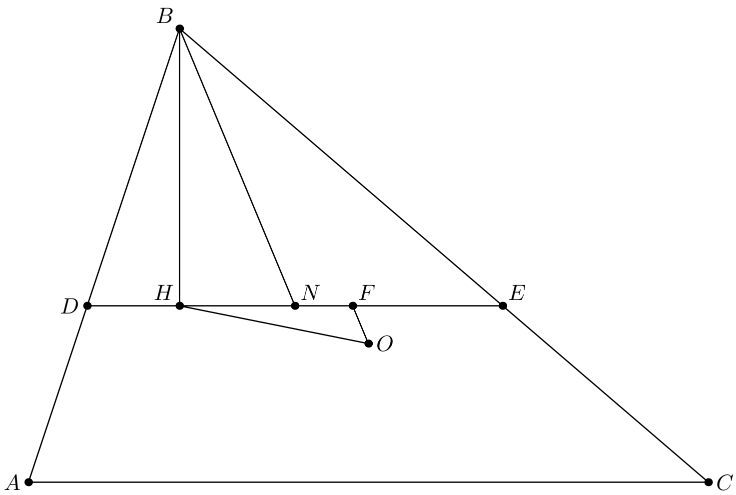

Fig. 9.

10. (8–9) (M.Evdokimov) An acute-angled triangle with one side equal to the altitude from the opposite vertex is cut from paper. Construct a point inside this triangle such that the square of the distance from it to one of the vertices equals the sum of the squares of distances to to the remaining two vertices. No instruments are available, it is allowed only to fold the paper and to mark the common points of folding lines.

Solution. Let the altitude CH of triangle ABC be equal to he side AB, $AC > BC$. Then for any point X lying on the altitude from A we have $XC^2 - XB^2 = AC^2 - AB^2 = AH^2$. Therefore it is sufficient to construct the point on this altitude such that $AX = AH$.

Fold the triangle along the lines passing through A and C, perpendicular to BC and AB respectively, mark the point H. Now fold in such a way that AB coincides with the altitude from A and mark the point X coinciding with H (fig. 10).

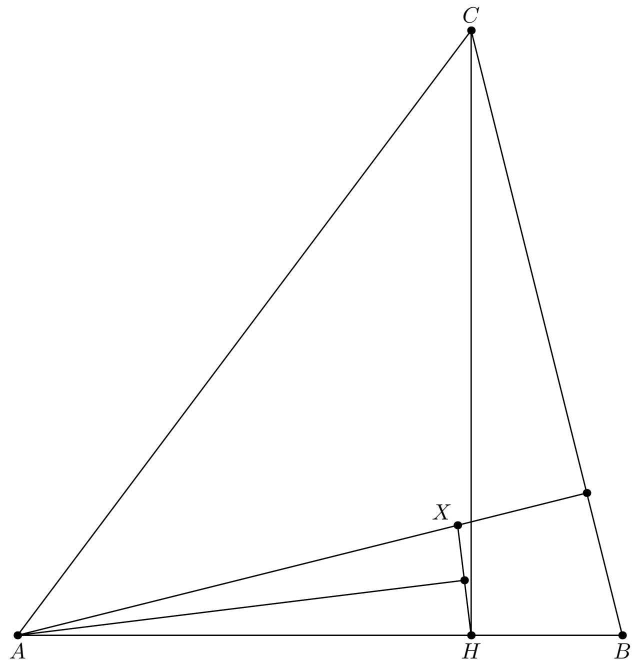

Fig. 10.

11. (8–10) (F.Nilov) A point X is the origin of three rays such that the angle between any two of them equals 120°. Let w be an arbitrary circle with radius R such that X lies inside it, and A, B, C be the common points of the rays with this circle. Find $\max(XA+XB+XC)$.

    Solution. The assumption yields that X is the Torricelli point of triangle ABC. Therefore $XA+XB+XC \leq OA+OB+OC = 3R$, where O is the center of the circle. The equality is obtained when O coincides with X.

12. (8–10) (L.Shatunov) Circles ω_1 and ω_2 are given. Let M be the midpoint of the segment joining their centers, X, Y be arbitrary points on ω_1, ω_2 respectively such that $MX = MY$. Find the locus of the midpoints of segments XY.

    Answer. One or two segments perpendicular to the line joining the centers of the circles.

    Solution. Let O_1, O_2 be the centers of the given circles, r_1, r_2 be their radii, and $O_1O_2 = 2d$. Constructing the altitude XH of triangle XMO_1 we obtain that $MH^2 - O_1H^2 = (MH+HO_1)(MH-HO_1) = XM^2 - r_1^2$. Since one of multipliers equals d, the second one equals $(XM^2 - r_1^2)/d$ and $MH = (XM^2 + d^2 - r_1^2)/2d$. Similarly the distance between M and the projection of Y to O_1O_2 equals $(XM^2 + d^2 - r_2^2)/2d$, therefore the projection of the midpoint of XY to O_1O_2 do not depend on XM. If ω_1 and ω_2 do not intersect, the required locus contains two segments symmetric with respect to O_1O_2, their endpoints correspond to the maximal and the minimal values of XM (fig. 12). In the opposite case we obtain one segment.

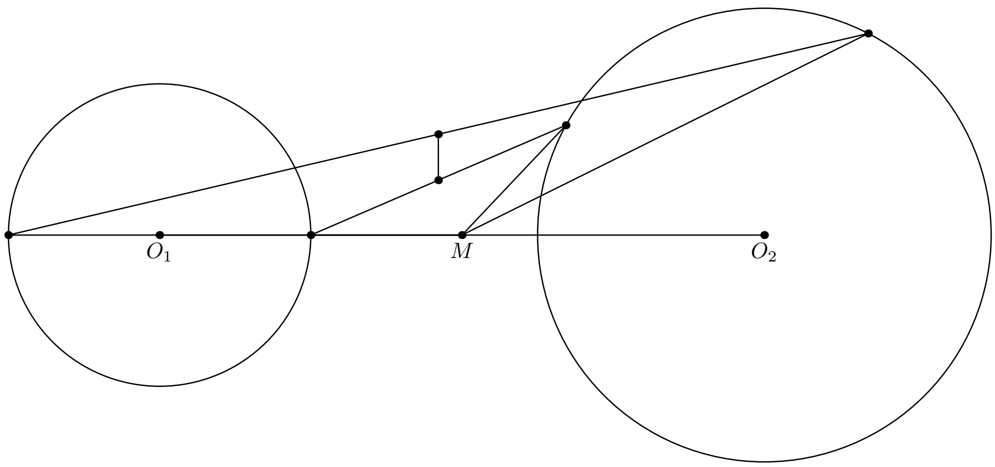

Fig. 12.

13. (8–11) (B.Frenkin) Each two opposite sides of a convex $2n$-gon are parallel. (Two sides are opposite if one passes $n-1$ other sides moving from one side to another along the borderline of the $2n$-gon.) The pair of opposite sides is called *regular* if there exists a common perpendicular to them such that its endpoints lie on the sides and not on their extensions. Which is the minimal possible number of regular pairs?

Answer. 1.

Example. Take the parallelogram $ABCD$ such that the projection of segment BC to the line AD do not intersect the segment AD. Choose the points B_1, B_2 on the sides AB, BC and the points D_1, D_2 on the sides CD, AD in such a way that $B_1B_2 \parallel D_1D_2$ and the projections of segments B_1B_2, D_1D_2 to a parallel line do not intersect (fig. 13). Similarly choose the points on the segments B_1B_2, B_2C, D_1D_2, D_2A etc. We obtain a $2n$-gon with unique regular pair AB_1, CD_1.

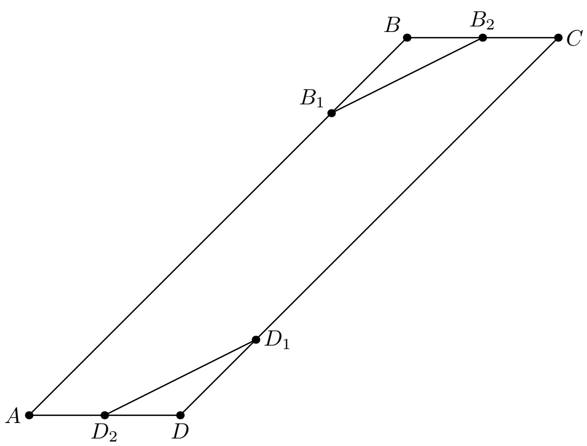

Fig. 13.

Estimate. Let $A_{2n}A_1$, A_nA_{n+1} be the pair of sidelines with minimal distance between them. Suppose that the projections of these sides to a parallel line do not intersect. Then we can suppose that $\angle A_{2n-1}A_{2n}A_1 > \pi/2$. Extend the side $A_{2n-1}A_{2n}$, $A_{n-1}A_n$ until their meeting points B, C with the lines A_nA_{n+1}, $A_{2n}A_1$ respectively. The perpendicular from A_{2n} to A_nB meets the side A_nC of parallelogram $A_{2n}CA_nB$, Therefore the distance from A_{2n} to A_nB is grater than the distance to A_nC, which contradicts to the definition of the pair $A_{2n}A_1$, A_nA_{n+1}.

14. (9–11) (L.Shatunov) A point D lies inside a triangle ABC on the bisector of angle B. Let ω_1 and ω_2 be the circles touching AD and CD at D and passing through B; P and Q be the common points of ω_1 and ω_2 with the circumcircle of ABC distinct from B. Prove that the circumcircles of the triangles PQD and ACD are tangent.

    Solution. Take an inversion centered at D with radius DB, let A', C' P', Q' be the images of A, C, P, Q respectively. Then $\angle DC'B = \angle CBD = \angle ABD = \angle DA'B$. Also ω_1 and ω_2 are transformed to the lines passing through B and parallel to $A'D$, $C'D$ respectively. Therefore $\angle P'BA' = \angle Q'BC'$ and $A'P'Q'C'$ is an isosceles trapezoid (fig. 14). Now the inversion maps the parallel lines $P'Q'$ and $A'C'$ to the circles touching at D.

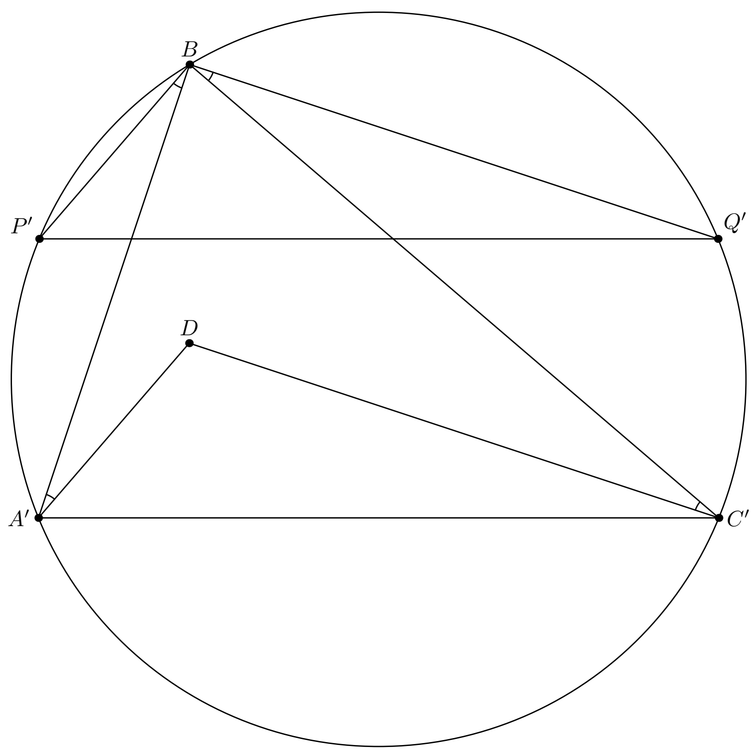

Fig. 14.

15. (9–11) (A.Zaslavsky) A point C lies on the bisector of an acute angle with vertex S. Let P, Q be the projections of C to the sidelines of the angle. The circle centered at C with radius PQ meets the sidelines at points A and B such that $SA \neq SB$. Prove that the circle with center A touching SB and the circle with center B touching SA are tangent.

    Solution. Since SC is the bisector of angle ASB, $AC = BC$, and $SA \neq SB$, we obtain that S, A, B, C are concyclic. Hence $\angle CAB = \angle CSB = \angle CPQ$, i.e. the triangles CPQ

and CAB are similar. Therefore $AB : PQ = PQ : PC = 2\cos\angle CSP$. Also $AP = BQ$, thus $SA + SB = 2SP$ and the sum of distances from A and B to the opposite sides of the angle equals $2SP\sin\angle ASB = 2PC\cos^2\angle ASC = AB$, which is equivalent to the required assertion (fig. 15).

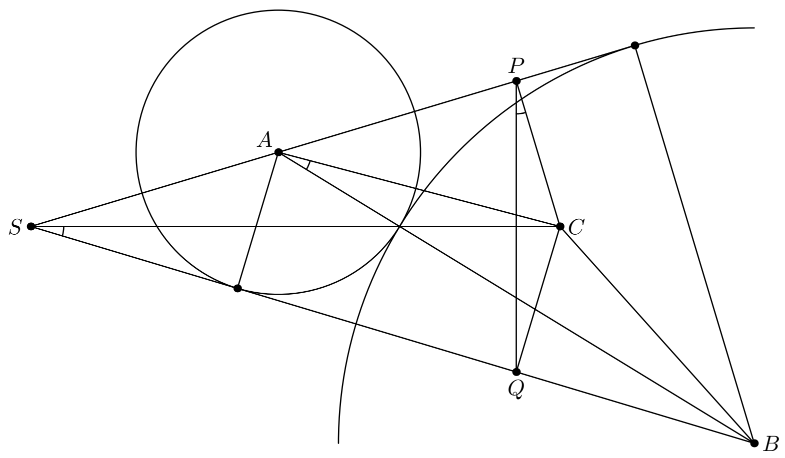

Fig. 15.

16. (9–11) (A.Zaslavsky) The Feuerbach point of a scalene triangle lies on one of its bisectors. Prove that it bisects the segment between the corresponding vertex and the incenter.

    First solution. Let the Feuerbach point F of triangle ABC lie on the bisector of angle C. Then the center E of the nine-points circle bisecting the segment between the orthocenter H and the circumcenter O also lies on this bisector. Thus CE bisects the angle OCH, Therefore O and H are symmetric with respect to the bisector, and $CO = CH = 2CO|\cos\angle C|$, i.e. the angle C equals to $\pi/3$ or $2\pi/3$. But in the second case O and H are symmetric with respect to external bisector of angle C. Hence $\angle C = \pi/3$, and $CF = r = CI\sin(\angle C/2) = CI/2$.

    Second solution. It is known that F if the center of the equilateral hyperbola passing through A, B, C, I. If the line CI passes through the center of this hyperbola, C and I are symmetric with respect to F.

17. (9–11) (P.Puchkov, E.Utkin) Let O, I be the circumcenter and the incenter of an acute-angled scalene triangle ABC; D, E, F be the touching points of its excircle with the side BC and the extensions of AC, AB respectively. Prove that if the orthocenter of the triangle DEF lies on the circumcircle of ABC, then it is symmetric to the midpoint of the arc BC with respect to OI.

    Solution. Let D', E', F' be the second common points of the altitudes of triangle DEF with its circumcircle. Then the altitudes are the bisectors of triangle $D'E'F'$ (one internal and two external), therefore they are parallel to the corresponding bisectors of triangle ABC. Hence the sidelines of these triangles are also parallel, i.e. the triangles are homothetic. This homothety maps O and the excenter I_A to I_A and the orthocenter H of triangle DEF respectively, therefore H lies on the line I_AO, and $I_AH : OI_A = r_A : R$. From this and the equalities $OI_A = R^2 + 2Rr_A$, $OH = R$ we obtain that $OI_A = 2R$. Also it is known that the midpoint W of the arc BC bisects the segment II_A. Since the medial line WM of triangle OII_A is perppendicular to HW, we obtain that $OI \perp HW$ (fig. 17), which yields the required assertion.

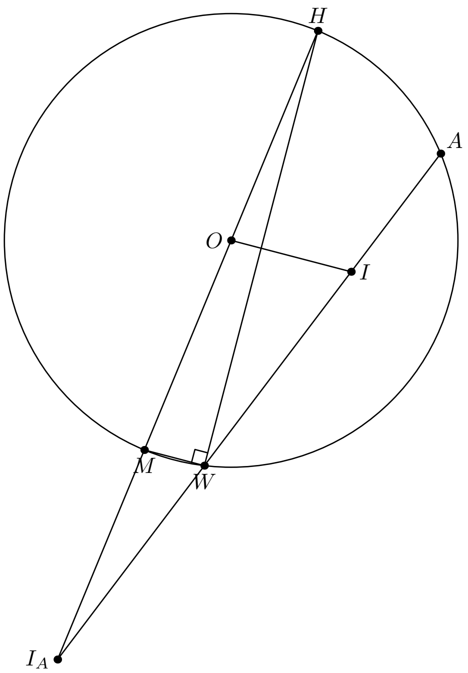

Fig. 17.

18. (9–11) (I.Kukharchuk) Let $ABCD$ be a quadrilateral such that the excircles ω_1 and ω_2 of triangles ABC and BCD touching their sides AB and BD respectively touch the extension of BC at the same point P. The segment AD meets ω_2 at point Q, and the line AD meets ω_1 at R and S. Prove that one of angles RPQ and SPQ is right.

    Solution. We have to prove that AD passes through the center of internal homothety mapping ω_1 to ω_2. Let ω_1 touch BA and AC at E and G respectively. Let ω_2 touch BD and CD at F and L respectively.

    Applying the Menelaos theorem to the triangles $ABD <$ ACD and the lines EF, GL respectively we obtain that EF and GL meet on AD, on the other hand EF passes through the homothety center, because EF meets ω_1 in the point such the tangent at it is parallel to BD. Similarly GL passes through this homothety center (fig. 18).

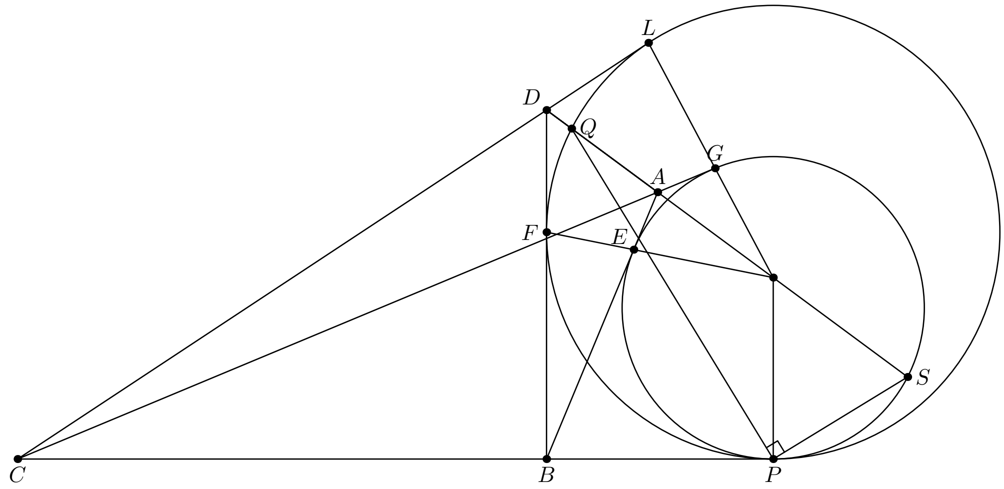

Fig. 18.

19. (10–11) (S.Kuznetsov) Let I be the incenter of a triangle ABC; A', B', C' be the orthocenters of the triangles BIC, AIC, AIB; M_a, M_b, M_c be the midpoints of BC, CA, AB, and S_a, S_b, S_c be the midpoints of AA', BB', CC'. Prove that M_aS_a, M_bS_b, M_cS_c concur.

Solution. The reflection of C' about M_c is opposite to I on the circle IAB, i.e. coincide with the excenter I_C of triangle ABC. Hence the medial line S_cM_c of triangle $C'II_C$ is parallel to CI and passes through the incenter of triangle $M_aM_bM_c$ (fig. 19). Two remaining lines also pass through this point.

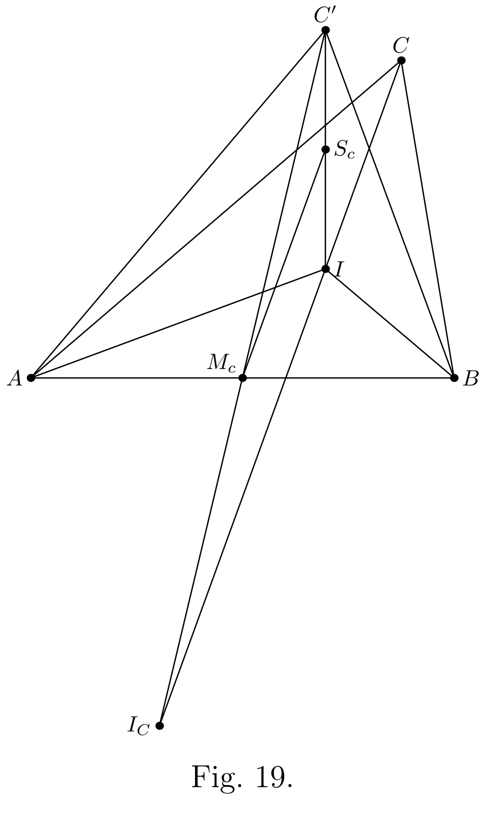

Fig. 19.

20. (10–11) (F.Ivlev) Let H be the orthocenter of a triangle ABC, and M, N be the midpoints of segments BC, AH respectively. The perpendicular from N to MH meets BC at point A'. Points B' and C' are defined similarly. Prove that A', B', C' are collinear.

Solution. Since MN is a diameter of the nine-points circle, the projection of N to MH lies on this circle (fig. 20). Hence A' on the polar of H with respect to the nine-points circle. The points B', C' also lie on this line.

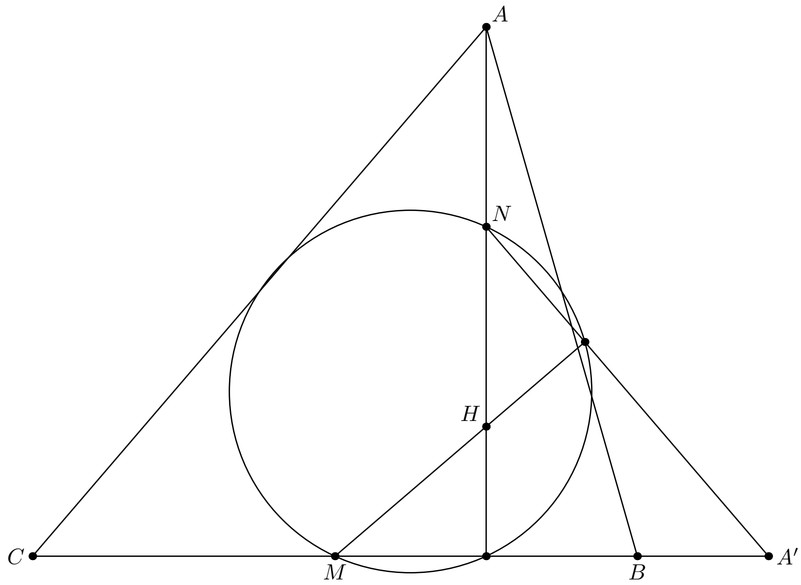

Fig. 20.

21. (10–11) (G.Galyapin) Let P be a point inside a quadrilateral $ABCD$ such that $\angle APB + \angle CPD = 180°$. Points P_a, P_b, P_c, P_d are isogonally conjugated to P with respect to the triangles BCD, CDA, DAB, ABC respectively. Prove that the diagonals of the quadrilaterals $ABCD$ and $P_aP_bP_cP_d$ concur.

Solution. Since $\angle APB + \angle CPD = 180°$, there exists a point Q isogonally conjugated to P with respect to $ABCD$. Then P_c, P_a lie on AQ, CQ respectively in such a way that $\angle P_cBQ = \angle DBC$, $\angle P_aBQ = \angle DBA$ (fig. 21). Hence $AP_c : P_cQ = AB \sin \angle ABP_c : BQ \sin \angle P_cBQ$, $QP_a : P_aC = BQ \sin \angle QBP_a : BC \sin \angle P_aBC$, and by the Menelaos theorem P_aP_c divides AC in ratio $PA \sin \angle ABD : PC \sin \angle DBC$. Therefore this line passes through the common point L of AC and BD (fig. 21). Similarly P_bP_d passes through L.

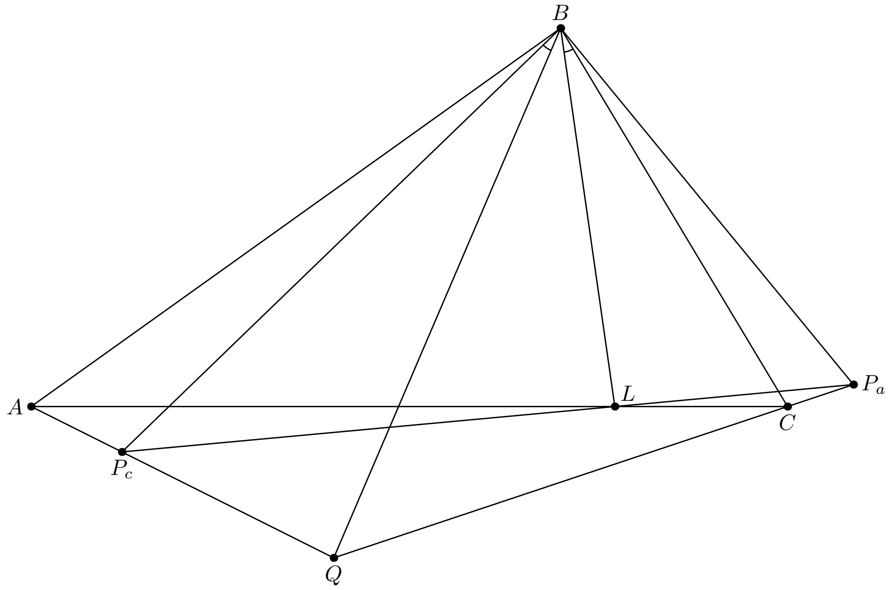

Fig. 21.

22. (10–11) (A.Zaslavsky) A circle and an ellipse with foci F_1, F_2 lying inside it are given. Construct a chord AB of the circle touching the ellipse and such that AF_1F_2B is a cyclic quadrilateral.

First solution.

Lemma. Let AB be an arbitrary chord touching the ellipse. Then the locus of circumcenters of triangles ABF_1 is a circle.

Proof. Let O, R be the center and the radius of the given circle; O' be the circumcenter of triangle ABF_1; H be the projection of F_1 to AB (fig. 22). It is clear that $OO' \parallel F_1H$. Applying the cosines law to the triangles $O'OA$ and $O'OF_1$ we obtain

$$O'F_1^2 = O'O^2 + OF_1^2 - 2OO' \cdot OF_1 \cos \angle O'OF_1, \quad O'A^2 = O'O^2 + R^2 - 2O'O \cdot OA \cos \angle O'OA.$$

Substracting from the first equality the second one we obtain

$$R^2 - OF_1^2 = 2O'O(OA \cos \angle O'OA - OF_1 \cos \angle O'OF_1).$$

We have in the parenthesis the difference of projections of segments OA and OF_1 to OO', which equals F_1H. Thus the product $OO' \cdot F_1H$ does not depend on AB.

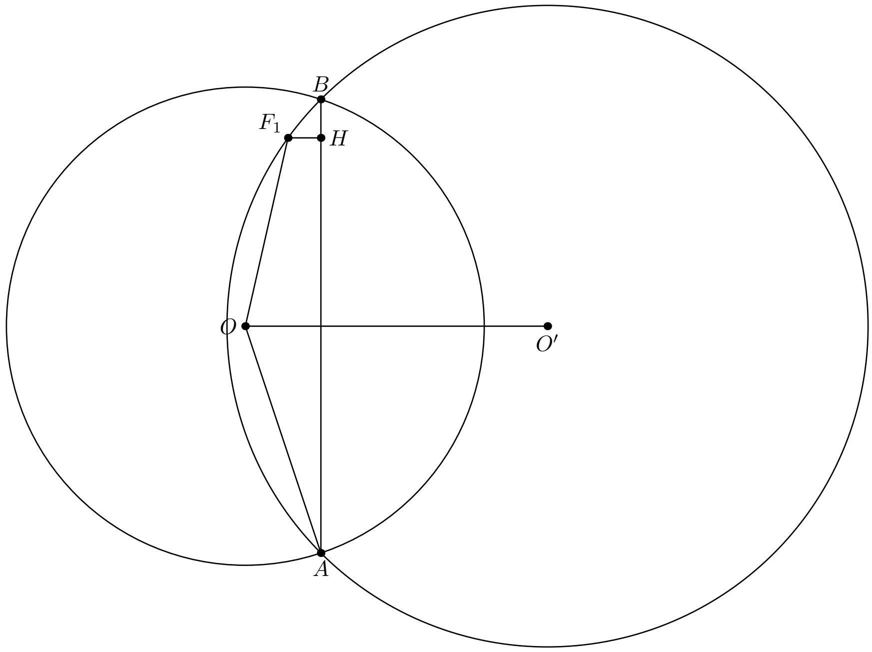

Fig. 22.

Let E be the pole of AB with respect to some circle centered at F_1. Then $F_1E \parallel OO'$ and the ratio $OO' : F_1E$ does not depend on AB because the products $F_1H \cdot OO'$ and $F_1H \cdot F_1E$ do not depend on AB. Thus $O'E$ meets OF_1 at the fixed point X. Hence the locus of O' is homothetic with respect to X to the polar map of ellipse with respect to F_1 which is a circle.

Return to the problem. Since the tangents to the ellipse can be constructed by a compass and a ruler, we can construct the loci of circumcenters of triangles ABF_1 and ABF_2. Their common point is the center of the required circle.

Second solution. Let D and E be the common points of the given circle ω and the line F_1F_2. Let S be a point on F_1F_2 such that $SD \cdot SE = SF_1 \cdot SF_2$. To construct this point take any circle passing through F_1 and F_2 and intersecting ω. The point S is the common point of the radical axis of two circles with F_1F_2. Let Ω be the circle having the great axis of the ellipse as a diameter, S' be the reflection of S about Ω, C be a point on the ellipse such that $CS' \perp F_1F_2$, A and B be the common points of ω and CS. Then CS touches the ellipse, and the quadrilateral AF_1F_2B is cyclic.

23. (10–11) (N.Spivak) Let us say that a subset M of the plane contains a hole if there exists a disc not contained in M, but contained inside some polygon with the boundary lying in M.

Can the plane be presented as a union of n convex sets such that the union of any $n-1$ from them contains a hole?

Answer. Yes.

Solution. Let $n = 6$; the set M_0 be a regular pentagon $A_1A_2A_3A_4A_5$, and the sets M_i, $i = 1, \dots, 5$ be the semiplanes bounded by the lines A_iA_{i+1} $(A_{i+5} = A_i)$, not containing the pentagon. Then the union of M_i is the whole plane, the union of $M_i, \dots, M_5$ is the plane without the pentagon M_0, and the union of five sets excepting M_i is the plane without the triangle formed by the lines $A_{i-1}A_i$, A_iA_{i+1}, and $A_{i+1}A_{i+2}$.

Remark. Jury does know the examples for $n < 6$.

24. (11) (S.Arutyunyan) The insphere of a tetrahedron $ABCD$ touches the faces ABC, BCD, CDA, DAB at D', A', B', C' respectively. Denote by S_{AB} the square of the triangle $AC'B$. Define similarly S_{AC}, S_{BC}, S_{AD}, S_{BD}, S_{CD}. Prove that there exists a triangle with sidelengths $\sqrt{S_{AB}S_{CD}}$, $\sqrt{S_{AC}S_{BD}}$, $\sqrt{S_{AD}S_{BC}}$.

    Solution Use the known equality $\angle AC'B = \angle AD'B = \angle CA'D = \angle CB'D$ and three similar to it. Denote by a, b, c, d and α, β, γ the lengths of the tangents to the insphere from A, B, C, D, and the angles $BD'C$, $CD'A$, $AD'B$ respectively. Then

$$S_{AB}S_{CD} = \frac{abcd\sin^2\gamma}{4}.$$

    Since the angles α, β, γ are less than π, we have to prove the existence of a triangle with sidelines $\sin\alpha$, $\sin\beta$, $\sin\gamma$ which clearly follows from the equality $\alpha + \beta + \gamma = 2\pi$. For example we can take a triangle formed by the lines perpendicular to $D'A$, $D'B$, $D'C$.