IX GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND

Below is the list of problems for the first (correspondence) round of the IX Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of 8–11 grades (these are four elder grades in Russian school). In the list below, each problem is indicated by the numbers of school grades, for which it is intended. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

The solutions for the problems (in Russian or in English) must be contained in pdf, doc or jpg files. Your work must be sent not later than on April 1, 2013, by e-mail to geomolymp@mccme.ru. Please, follow a few simple rules:

1. Each student sends his work in a separate message (with delivery notification). The size of the message must not exceed 10 Mb.

2. If your work consists of several files, send it as an archive.

3. If the size of your message exceeds 10 Mb divide it into several messages.

4. In the subject of the message write "The work for Sharygin olympiad", and present the following personal data in the body of your message:

- last name;

- all other names;

- E-mail, phone number, post address;

- the current number of your grade at school;

- the number of the last grade at your school;

- the number and/or the name and the mail address of your school;

- full names of your teachers in mathematics at school and/or of instructors of your extra math classes (if you attend additional math classes after school).

If you have no e-mail access, please send your work by ordinary mail to the following address: *Russia, 119002, Moscow, Bolshoy Vlasyevsky per., 11. Olympiad in honour of Sharygin.* In the title page, write down your personal information indicated in the item 4 above.

We recommend you to write your work on the special blanks that can be found on www.blank.geomolymp.mccme.ru. This will provide quick and qualitative examination of your work. If you type the work using a computer then please receive a blank on the site above and copy its bar code (a square with a pattern in the right upper corner of the blank) into your work as a picture. If your work consists of several files, copy this bar code into all files, this enables to identify the author of the work.

In your work, please start the solution for each problem in a new page. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all significant arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

Winners of the correspondence round, the students of three grades before the last grade, will be invited to the final round in Summer 2013 in the city of Dubna, in Moscow region. (

For instance, if the last grade is 12, then we invite winners from 9, 10, and 11 grade.) Winners of the correspondence round, the students of the last grade, will be awarded by diplomas of the Olympiad. The list of the winners will be published on www.geometry.ru at the end of May 2013. If you want to know your detailed results, please use e-mail.

1. (8) Let ABC be an isosceles triangle with $AB = BC$. Point E lies on the side AB, and ED is the perpendicular from E to BC. It is known that $AE = DE$. Find $\angle DAC$.

2. (8) Let ABC be an isosceles triangle ($AC = BC$) with $\angle C = 20^\circ$. The bisectors of angles A and B meet the opposite sides at points A_1 and B_1 respectively. Prove that the triangle A_1OB_1 (where O is the circumcenter of ABC) is regular.

3. (8) Let ABC be a right-angled triangle ($\angle B = 90^\circ$). The excircle inscribed into the angle A touches the extensions of the sides AB, AC at points A_1, A_2 respectively; points C_1, C_2 are defined similarly. Prove that the perpendiculars from A, B, C to C_1C_2, A_1C_1, A_1A_2 respectively, concur.

4. (8) Let ABC be a nonisosceles triangle. Point O is its circumcenter, and point K is the center of the circumcircle w of triangle BCO. The altitude of ABC from A meets w at a point P. The line PK intersects the circumcircle of ABC at points E and F. Prove that one of the segments EP and FP is equal to the segment PA.

5. (8) Four segments drawn from a given point inside a convex quadrilateral to its vertices, split the quadrilateral into four equal triangles. Can we assert that this quadrilateral is a rhombus?

6. (8–9) Diagonals AC and BD of a trapezoid $ABCD$ meet at point P. The circumcircles of triangles ABP and CDP intersect the line AD for the second time at points X and Y respectively. Let M be the midpoint of segment XY. Prove that $BM = CM$.

7. (8–9) Let BD be a bisector of triangle ABC. Points I_a, I_c are the incenters of triangles ABD, CBD respectively. The line I_aI_c meets AC in point Q. Prove that $\angle DBQ = 90^\circ$.

8. (8–9) Let X be an arbitrary point inside the circumcircle of a triangle ABC. The lines BX and CX meet that circumcircle in points K and L respectively. The line LK intersects BA and AC at points E and F respectively. Find the locus of points X such that the circumcircles of triangles AFK and AEL touch.

9. (8–9) Let T_1 and T_2 be the points of tangency of the excircles of a triangle ABC with its sides BC and AC respectively. It is known that the reflection of the incenter of ABC across the midpoint of AB lies on the circumcircle of triangle CT_1T_2. Find $\angle BCA$.

10. (8–9) The incircle of triangle ABC touches the side AB at point C'; the incircle of triangle ACC' touches the sides AB and AC at points C_1, B_1; the incircle of triangle BCC' touches the sides AB and BC at points C_2, A_2. Prove that the lines B_1C_1, A_2C_2, and CC' concur.

11. (8–9) a) Let $ABCD$ be a convex quadrilateral and $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triagles ABC, BCD, CDA, DAB. Can the inequality $r_4 > 2r_3$ hold?

    b)The diagonals of a convex quadrilateral $ABCD$ meet in point E. Let $r_1 \le r_2 \le r_3 \le r_4$ be the radii of the incircles of triangles ABE, BCE, CDE, DAE. Can the inequality $r_2 > 2r_1$ hold?

12. (8–11) On each side of a triangle ABC, two distinct points are marked. It is known that these points are the feet of the altitudes and of the bisectors.

    a) Using only a ruler determine which points are the feet of the altitudes and which points are the feet of the bisectors.

    b) Solve p.a) drawing only three lines.

13. (9–10) Let A_1 and C_1 be the tangency points of the incircle of triangle ABC with BC and AB respectively, A' and C' be the tangency points of the excircle inscribed into the angle B with the extensions of BC and AB respectively. Prove that the orthocenter H of triangle ABC lies on A_1C_1 if and only if the lines $A'C_1$ and BA are orthogonal.

14. (9–11) Let M, N be the midpoints of diagonals AC, BD of a right-angled trapezoid $ABCD$ ($\angle A = \angle D = 90°$). The circumcircles of triangles ABN, CDM meet the line BC in points Q, R. Prove that the distances from Q, R to the midpoint of MN are equal.

15. (9–11) a) Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are inscribed into triangle ABC so that $C_1A_1 \perp BC$, $A_1B_1 \perp CA$, $B_1C_1 \perp AB$, $B_2A_2 \perp BC$, $C_2B_2 \perp CA$, $A_2C_2 \perp AB$. Prove that these triangles are equal.

    b) Points $A_1, B_1, C_1, A_2, B_2, C_2$ lie inside a triangle ABC so that A_1 is on segment AB_1, B_1 is on segment BC_1, C_1 is on segment CA_1, A_2 is on segment AC_2, B_2 is on segment BA_2, C_2 is on segment CB_2, and the angles $BAA_1, CBB_1, ACC_1, CAA_2, ABB_2, BCC_2$ are equal. Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are equal.

16. (9–11) The incircle of triangle ABC touches BC, CA, AB at points A', B', C' respectively. The perpendicular from the incenter I to the median from vertex C meets the line $A'B'$ in point K. Prove that $CK \parallel AB$.

17. (9–11) An acute angle between the diagonals of a cyclic quadrilateral is equal to ϕ. Prove that an acute angle between the diagonals of any other quadrilateral having the same sidelengths is smaller than ϕ.

18. (9–11) Let AD be a bisector of triangle ABC. Points M and N are the projections of B and C respectively to AD. The circle with diameter MN intersects BC at points X and Y. Prove that $\angle BAX = \angle CAY$.

19. (10–11) a) The incircle of a triangle ABC touches AC and AB at points B_0 and C_0 respectively. The bisectors of angles B and C meet the perpendicular bisector to the bisector AL in points Q and P respectively. Prove that the lines PC_0, QB_0, and BC concur.

    b) Let AL be the bisector of a triangle ABC. Points O_1 and O_2 are the circumcenters of triangles ABL and ACL respectively. Points B_1 and C_1 are the projections of C and B to the bisectors of angles B and C respectively. Prove that the lines O_1C_1, O_2B_1, and BC concur.

    c) Prove that two points obtained in pp. a) and b) coincide.

20. (10–11) Let C_1 be an arbitrary point on the side AB of triangle ABC. Points A_1 and B_1 on the rays BC and AC are such that $\angle AC_1B_1 = \angle BC_1A_1 = \angle ACB$. The lines AA_1 and BB_1 meet in point C_2. Prove that all the lines C_1C_2 have a common point.

21. (10–11) Let A be a point inside a circle ω. One of two lines drawn through A intersects ω at points B and C, the second one intersects it at points D and E (D lies between A and E). The line passing through D and parallel to BC meets ω for the second time at point F, and the line AF meets ω at point T. Let M be the common point of the lines ET and BC, and N be the reflection of A across M. Prove that the circumcircle of triangle DEN passes through the midpoint of segment BC.

22. (10–11) The common perpendiculars to the opposite sidelines of a nonplanar quadrilateral are mutually orthogonal. Prove that they intersect.

23. (10–11) Two convex polytopes A and B do not intersect. The polytope A has exactly 2012 planes of symmetry. What is the maximal number of symmetry planes of the union of A and B, if B has a) 2012, b) 2013 symmetry planes?

    c) What is the answer to the question of p.b), if the symmetry planes are replaced by the symmetry axes?