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A Generalization of the Kiepert Hyperbola

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Abstract. Consider an arbitrary point P in the plane of triangle ABC with cevian triangle $A_1B_1C_1$. Erecting similar isosceles triangles on the segments BA_1, CA_1, CB_1, AB_1, AC_1, BC_1, we get six apices. If the apices of the two isosceles triangles with bases BA_1 and CA_1 are connected by a line, and the two similar lines for B_1 and C_1 are drawn, then these three lines form a new triangle, which is perspective to triangle ABC. For fixed P and varying base angle of the isosceles triangles, the perspector draws a hyperbola. Some properties of this hyperbola are studied in the paper.

1. Introduction

We consider the following configuration. Let P be a point in the plane of a triangle ABC, and AA_1, BB_1 and CC_1 be the three cevians of P. For an arbitrary nonzero angle φ satisfying $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, we erect two isosceles triangles BA_bA_1 and CA_cA_1 with the bases BA_1 and A_1C and base angle φ, both externally to triangle ABC if $\varphi > 0$, and internally otherwise. The same construction also gives the points B_c, B_a, C_a, C_b, with isosceles triangles all with base angle φ. This configuration depends on triangle ABC, the point P and $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}$.

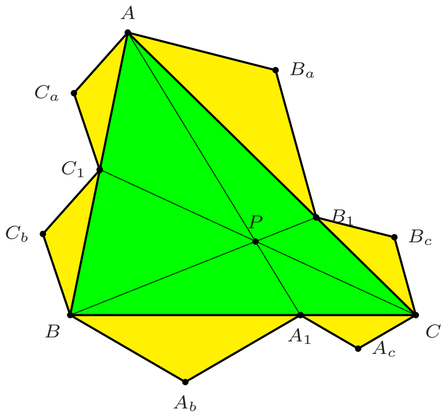

Figure 1.

We study an interesting locus problem associated with this configuration.

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2. The coordinates of the vertices

In this paper we work with homogeneous barycentric coordinates, and make use of John H. Conway's notations. See [1] for some basic properties of the Conway symbols. We begin by calculating the barycentric coordinates of the apices of our isosceles triangles. Let $(u : v : w)$ be the homogeneous barycentric coordinates of the point P.

Proposition 1. *The apices of the isosceles triangles on BA_1 and A_1C are the points*

$$A_b =(-a^2w : 2S_\varphi v + (S_C + S_\varphi)w : (S_B + S_\varphi)w), \tag{1}$$

$$A_c =(-a^2v : (S_C + S_\varphi)v : 2S_\varphi w + (S_B + S_\varphi)v). \tag{2}$$

Proof. Let A_φ be the apex of the isosceles triangle with base BC and base angle φ. It is well known that the point A_φ has the coordinates $(-a^2 : S_C + S_\varphi : S_B + S_\varphi)$. The line A_bA_1 is parallel to the line $A_\varphi C$; hence, using directed segments, we have $\frac{BA_b}{A_bA_\varphi} = \frac{BA_1}{A_1C} = \frac{w}{v}$, so that (identifying every point with the vector to the point from an arbitrarily chosen origin),

$$A_b = \frac{vB + wA_\varphi}{v + w} = \left(-\frac{a^2w}{2S_\varphi} : \frac{(S_C + S_\varphi)w}{2S_\varphi} + v : \frac{(S_B + S_\varphi)w}{2S_\varphi}\right).$$

Here, we have used the fact that the sum of the coordinates of the point A_φ is $-a^2 + (S_B + S_C) + 2S_\varphi = 2S_\varphi$.

This yields the coordinates of A_b given in (1) above. Similarly, A_c is as given in (2). The four remaining apices can be computed readily. □

Let $\mathcal{L}_a$ be the line joining the apices A_b and A_c. It is routine to compute the barycentric equation of the line $\mathcal{L}_a$.

Proposition 2. *The equation of the line $\mathcal{L}_a$ is*

$$(S_Bv^2 + S_Cw^2 + S_\varphi(v + w)^2)x + a^2w^2y + a^2v^2z = 0. \tag{3}$$

Proof. For $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}$, the equation of the line joining A_b and A_c is

$$\begin{vmatrix} x & y & z \\ -a^2w & 2S_\varphi v + (S_C + S_\varphi)w & (S_B + S_\varphi)w \\ -a^2v & (S_C + S_\varphi)v & 2S_\varphi w + (S_B + S_\varphi)v \end{vmatrix} = 0.$$

This simplifies into (3) above. □

Similarly, we define $\mathcal{L}_b$ and $\mathcal{L}_c$. Their equations can be easily written down:

$$b^2w^2x + (S_Cw^2 + S_Au^2 + S_\varphi(w + u)^2)y + b^2u^2z =0, \tag{4}$$

$$c^2v^2x + c^2u^2y + (S_Au^2 + S_Bv^2 + S_\varphi(u + v)^2)z =0. \tag{5}$$

3. The triangle formed by the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$

Consider the triangle bounded by the lines $\mathcal{L}_a$, $\mathcal{L}_b$ and $\mathcal{L}_c$. This has vertices

$$A_2 = \mathcal{L}_b \cap \mathcal{L}_c, \qquad B_2 = \mathcal{L}_c \cap \mathcal{L}_a, \qquad C_2 = \mathcal{L}_a \cap \mathcal{L}_b.$$

Theorem 3. *The triangle bounded by the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ is perspective with ABC. Their axis of perspectivity is the trilinear polar of the barycentric square of the point P.*

Proof. Let $A_0 = BC \cap \mathcal{L}_a$, $B_0 = CA \cap \mathcal{L}_b$, and $C_0 = AB \cap \mathcal{L}_c$. In homogeneous barycentric coordinates, these are the points

$$A_0 = (0 : -v^2 : w^2), \quad B_0 = (u^2 : 0 : -w^2), \quad C_0 = (-u^2 : v^2 : 0)$$

respectively, and are all on the line

$$\frac{x}{u^2} + \frac{y}{v^2} + \frac{z}{w^2} = 0. \tag{6}$$

It follows from the Desargues theorem that ABC and the triangle bounded by the lines $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ are perspective. Note that the axis of perspectivity (6) is the trilinear polar of the point $(u^2 : v^2 : w^2)$, the barycentric square of P.[1] It is independent of φ. $\square$

The perspector of the triangles, however, varies with φ. We work out its coordinates explicitly. The vertices of the triangle in question are

$$A_2 = \mathcal{L}_b \cap \mathcal{L}_c, \qquad B_2 = \mathcal{L}_c \cap \mathcal{L}_a, \qquad C_2 = \mathcal{L}_a \cap \mathcal{L}_b.$$

From (4) and (5), the line joining A_2 to A has equation

$$\begin{aligned} &c^2(v^2(S_C w^2 + S_A u^2 + 2S_\varphi(w+u)^2) - b^2w^2u^2)y \\ -\, &b^2(w^2(S_A u^2 + S_B v^2 + 2S_\varphi(u+v)^2) - c^2u^2v^2)z = 0. \end{aligned}$$

Similarly, by writing down the equations of the lines BB_2 and CC_2, we easily find the perspector of the triangles ABC and $A_2B_2C_2$.

Theorem 4. *For any point P and any angle φ, the perspector of the triangles ABC and $A_2B_2C_2$ is the point*

$$K_P(\varphi) = \left(\frac{a^2}{-S_B v^2(w^2-u^2) + S_C w^2(u^2-v^2) + u^2(v+w)^2 S_\varphi} : \cdots : \cdots \right) \tag{7}$$

Strictly speaking, the perspector $K_P(\varphi)$ is not defined in the cases $\varphi = 0$ and $\varphi = \frac{\pi}{2}$. However, in these two cases we can define the perspectors as the limits of the perspector when the angle approaches 0 and $\frac{\pi}{2}$, respectively. The coordinates of these limiting perspectors can be obtained from (7) by substituting $\varphi = 0$ and $\frac{\pi}{2}$ respectively.

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[1] If we take the harmonic conjugates A_0', B_0' and C_0' of the points A_0, B_0, C_0 with respect to the sides BC, CA, AB respectively, then the lines AA_0', BB_0' and CC_0' concur at the trilinear pole of the line $A_0B_0C_0$, which is the barycentric square of P. This gives an interesting construction of the barycentric square of a point. For another construction, see [2].

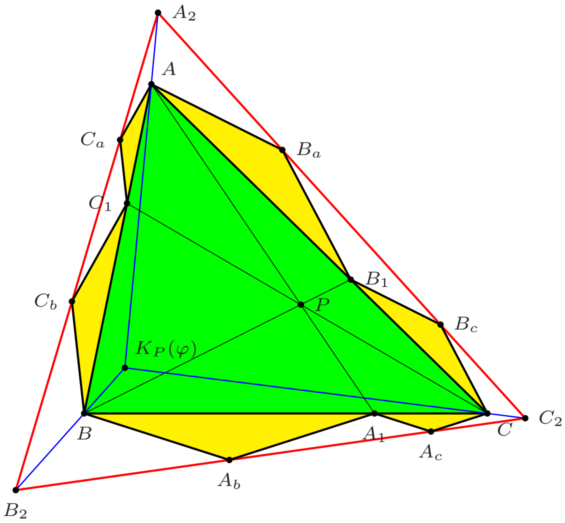

Figure 2.

$$K_P(0) = \left(\frac{a^2}{u^2(v+w)^2} : \frac{b^2}{v^2(w+u)^2} : \frac{c^2}{w^2(u+v)^2}\right),$$

$$K_P\left(\frac{\pi}{2}\right) = \left(\frac{a^2}{S_B\left(\frac{1}{w^2}-\frac{1}{u^2}\right) - S_C\left(\frac{1}{u^2}-\frac{1}{v^2}\right)} : \cdots : \cdots\right).$$

4. The locus of the perspector

From the coordinates of the perspector $K_P(\varphi)$ given in (7), it is clear that the point lies on the isogonal conjugate of the line joining the points

$$\begin{aligned} P_1 =& (u^2(v+w)^2 : v^2(w+u)^2 : w^2(u+v)^2), \\ P_2 =& (-S_B v^2(w^2-u^2) + S_C w^2(u^2-v^2) : \cdots : \cdots) \\ =& \left(S_B\left(\frac{1}{w^2}-\frac{1}{u^2}\right) - S_C\left(\frac{1}{u^2}-\frac{1}{v^2}\right) : \cdots : \cdots\right). \end{aligned}$$

Obviously P_1 is an interior point of triangle ABC. It is more interesting to note that P_2 is an infinite point, evidently of the line

$$S_A\left(\frac{1}{v^2}-\frac{1}{w^2}\right)x + S_B\left(\frac{1}{w^2}-\frac{1}{u^2}\right)y + S_C\left(\frac{1}{u^2}-\frac{1}{v^2}\right)z = 0. \tag{8}$$

Note that $\left(\frac{1}{v^2}-\frac{1}{w^2} : \frac{1}{w^2}-\frac{1}{u^2} : \frac{1}{u^2}-\frac{1}{v^2}\right)$ is also an infinite point, of the line (6). From (8), these two lines are orthogonal. See [3, p.52].

Theorem 5. *Let $P = (u : v : w)$. The locus $\mathcal{K}_P$ is the isogonal conjugate of the line through the point $(u^2(v+w)^2 : v^2(w+u)^2 : w^2(u+v)^2)$ perpendicular to the trilinear polar of $(u^2 : v^2 : w^2)$.*

If P is not the centroid [2] and if this line does not pass through any of the vertices of ABC or its antimedial triangle, then the locus $\mathcal{K}_P$ is a circum-hyperbola, [3] which is rectangular if and only if P lies on the quintic

$$\begin{aligned} &a^2v^2w^2(v-w)+b^2w^2u^2(w-u)+c^2u^2v^2(u-v)\\ =&uvw(u+v+w)((b^2-c^2)u+(c^2-a^2)v+(a^2-b^2)w). \end{aligned} \tag{9}$$

We shall study the degenerate case in §6 below.

5. Special cases

5.1. *The orthocenter.* If $P = H$, the orthocenter, $P_1 = (a^4 : b^4 : c^4)$ and the trilinear polar of $\left(\frac{1}{S_{AA}} : \frac{1}{S_{BB}} : \frac{1}{S_{CC}}\right)$ is the line $S_{AA}x + S_{BB}y + S_{CC}z = 0$. The perpendicular from P_1 to this line is the line

$$\frac{b^2-c^2}{a^2}x + \frac{c^2-a^2}{b^2}y + \frac{a^2-b^2}{c^2}z = 0,$$

which is clearly the Brocard axis OK. The locus $\mathcal{K}_H$ is therefore the Kiepert hyperbola $\mathcal{K}$. A typical point on $\mathcal{K}$ is the Kiepert perspector

$$K(\theta) = \left(\frac{1}{S_A+S_\theta} : \frac{1}{S_B+S_\theta} : \frac{1}{S_C+S_\theta}\right)$$

which is the perspector of the triangle of apices of isosceles triangles of base angles θ erected on the sides of triangle ABC.

Theorem 6. *$K_H(\varphi) = K(\theta)$ if and only if*

$$\cot\varphi(\cot\omega + \cot\theta) + \cot\theta\cot\omega + 1 = 0, \tag{10}$$

where ω is the Brocard angle of triangle ABC.

Proof. From (7),

$$K_H(\varphi) = \left(\frac{1}{S_{BC} - S_{AA} + a^2S_\varphi} : \cdots : \cdots\right).$$

This is the same as $K(\theta)$ if and only if

$$\begin{aligned} &((S_{CA} - S_{BB} + b^2S_\varphi)(S_{AB} - S_{CC} + c^2S_\varphi), \cdots, \cdots)\\ =&k((S_B+S_\theta)(S_C+S_\theta), \cdots, \cdots) \end{aligned}$$

for some k. These conditions are satisfied if and only if

$$k = (S_A + S_B + S_C + S_\varphi)^2,$$

and

$$S_\theta S_\varphi + (S_A+S_B+S_C)(S_\theta+S_\varphi) + S^2 = 0.$$

This latter condition translates into (10) above. □

[2] $K_G(\varphi) = K$, the symmedian point, for every φ.

[3] In fact, being the isogonal conjugate of the line P_1P_2, this is a circumconic. Since the line P_1P_2 intersects the circumcircle of triangle ABC (as P_1 is an interior point), it is a circum-hyperbola. The isogonal conjugate of the point P_2 is the fourth point of intersection of the circumscribed hyperbola with the circumcircle of triangle ABC.

Note that the relation (10) is symmetric in φ, ω, and θ. From this we obtain the following interesting corollary.

Corollary 7. *$K_H(\varphi) = K(\theta)$ if and only if $K_H(\theta) = K(\varphi)$.*

Here are some examples of corresponding φ and θ.

φ	$\frac{\pi}{4}$	$-\frac{\pi}{4}$	ω	$-\omega$
θ	$-\frac{\pi}{4}$	$\frac{\pi}{4}$	$-\arctan(\sin 2\omega)$	0

5.2. *The incenter.* If $P = I$, the incenter, we have

$$P_1 = (a^2(b+c)^2 : b^2(c+a)^2 : c^2(a+b)^2) = X_{1500}.$$

The point P_2 is the infinite point of the perpendicular to the Lemoine axis, namely,

$$X_{511} = (a^2(a^2(b^2+c^2) - (b^4+c^4)) : \cdots : \cdots),$$

the same as the case $P = H$. The hyperbola $\mathcal{K}_I$ is the circum-hyperbola through X_{98} and

$$X_{1509} = \left(\frac{1}{(b+c)^2} : \frac{1}{(c+a)^2} : \frac{1}{(a+b)^2}\right).$$

The center of the hyperbola is the point

$$((b-c)^2 f(a,b,c)g(a,b,c) : (c-a)^2 f(b,c,a)g(b,c,a) : (a-b)^2 f(c,a,b)g(c,a,b)),$$

where

$$\begin{aligned} f(a,b,c) =& a^5 - a^3(b^2+bc+c^2) - a^2(b+c)(b^2+c^2) - abc(b+c)^2 - b^2c^2(b+c), \\ g(a,b,c) =& a^5 - a^3(2b^2+bc+2c^2) - a^2(b+c)(2b^2+bc+2c^2) \\ & - a(b^4 - b^3c - 2b^2c^2 - bc^3 + c^4) - bc(b+c)^3. \end{aligned}$$

5.3. *The Gergonne point.* If P is the Gergonne point, P_1 is the symmedian point K and the infinite point of the perpendicular to the trilinear polar of

$$X_{279} = \left(\frac{1}{(b+c-a)^2} : \frac{1}{(c+a-b)^2} : \frac{1}{(a+b-c)^2}\right)$$

is

$$X_{517} = (a(a^2(b+c) - 2abc - (b+c)(b-c)^2) : \cdots : \cdots).$$

The hyperbola passes through the centroid and X_{104} and has center

$$(a^2(b-c)^2(a^3 - a^2(b+c) - a(b-c)^2 + (b+c)(b^2+c^2))^2 : \cdots : \cdots).$$

5.4. $P = X_{671}$. The point $X_{671} = \left(\frac{1}{2a^2-b^2-c^2} : \frac{1}{2b^2-c^2-a^2} : \frac{1}{2c^2-a^2-b^2}\right)$ lies on the quintic (9). It is the reflection of the centroid in the Kiepert center X_{115}. If $P = X_{671}$, the locus $\mathcal{K}_P$ is the rectangular hyperbola whose center is the point

$$((b^2-c^2)^2(2a^2-b^2-c^2)(a^4-b^4+b^2c^2-c^4) : \cdots : \cdots)$$

on the nine-point circle.

5.5. *P on a sideline.* If P is a point on a sideline of triangle ABC, say, BC, then $P_1 = (0 : 1 : 1)$ is the midpoint of BC, and $P_2 = (-a^2 : S_C : S_B)$ is the infinite point of the A-altitude. It follows that P_1P_2 is the perpendicular bisector of BC. Its isogonal conjugate is the circum-hyperbola whose center is the midpoint of BC. It also passes through the antipode of A in the circumcircle.

6. The degenerate case

The locus $\mathcal{K}_P$ is a circum-hyperbola if and only if the line P_1P_2 does not contain a vertex of the triangle. The equation of the line P_1P_2 is of the form

$$U(P)x + V(P)y + W(P)z = 0,$$

where

$$\begin{aligned} U(P) = {} & v^2(w+u)^2(w^2(u^2S_A + v^2S_B) - c^2u^2v^2) \\ & - w^2(u+v)^2(v^2(u^2S_A + w^2S_C) - b^2u^2w^2), \end{aligned}$$

and $V(P)$ and $W(P)$ are obtained from $U(P)$ by cyclic permutations of (u, v, w), (a, b, c) and (S_A, S_B, S_C). The locus of the perspectors $\mathcal{K}_P$ is degenerate (*i.e.*, it is not a hyperbola) if and only if at least one of the three coefficients $U(P)$, $V(P)$ and $W(P)$ in the equation of the line P_1P_2 is zero, *i.e.*, if the point P lies on at least one of the three curves of 8 degree defined by the equations $U(P) = 0$, $V(P) = 0$ and $W(P) = 0$. Each of these three curves contains the vertices of the triangle ABC, its centroid G, and also the vertices of the antimedial triangle $G_aG_bG_c$. Moreover, for any two of these three curves, the only real common points are these 7 points just listed. We conclude with the following observations.

- If P is one of the vertices A, B and C, then the locus $\mathcal{K}_P$ is not defined. It is possibly an isolated singularity of one or more of the curves $U(P) = 0$, $V(P) = 0$, and $W(P) = 0$.
- If P is a vertex of the antimedial triangle, then the locus $\mathcal{K}_P$ is the corresponding sideline of the triangle ABC. For example, $\mathcal{K}_{G_a} = BC$.
- If $P = G$, the centroid of triangle ABC, then $\mathcal{K}_P$ consists of one single point, the symmedian point K of triangle ABC.
- In all other degenerate cases, the hyperbola degenerates into a pair of lines, one of them being a sideline of the triangle, while the other one passes through the opposite vertex (but does not coincide with a sideline).

  If we put

  $$(u, v, w) = \left(\frac{1}{y+z-x}, \frac{1}{z+x-y}, \frac{1}{x+y-z}\right),$$

  the equation $U(P) = 0$ defines the quartic curve

  $$x = \frac{yz(S_By^2 - S_Cz^2)}{S_A(y-z)(y^2+z^2) + S_By^3 - S_Cz^3}$$

  with respect to the antimedial triangle of ABC. Figure 3 shows an example of these curves in which the vertex B is an isolated singularity of the curves $U(P) = 0$ and $W(P) = 0$.

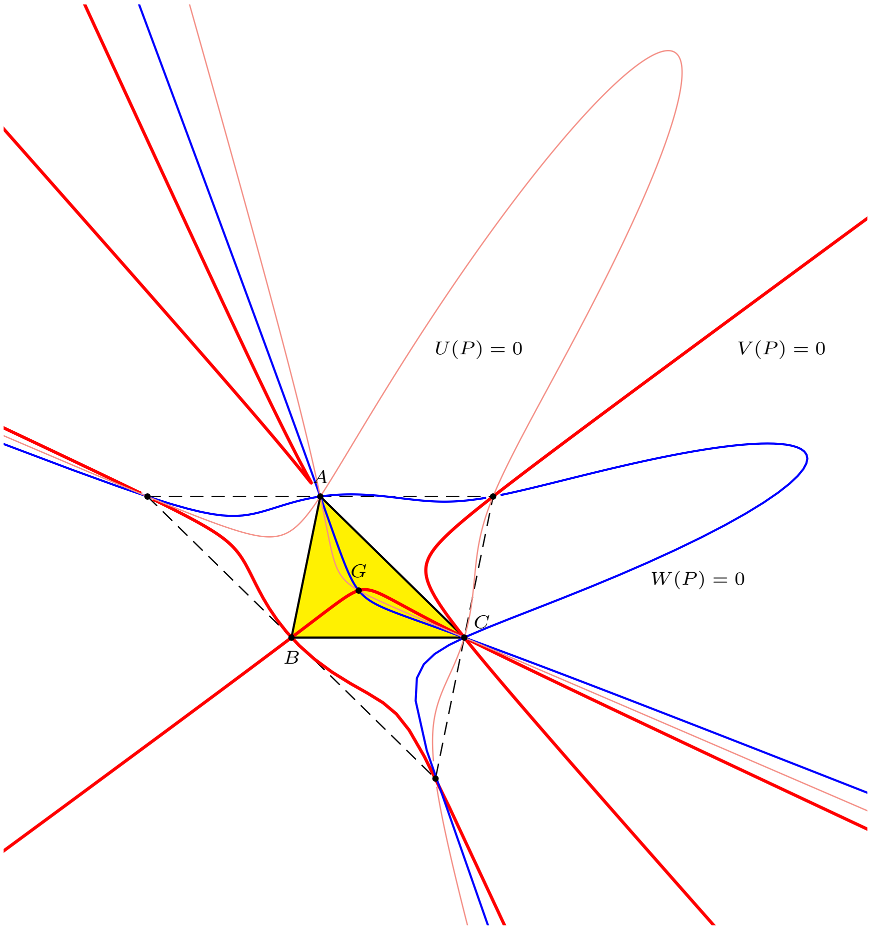

Figure 3.

References

[1] F. M. van Lamoen and P. Yiu, The Kiepert pencil of Kiepert hyperbolas, *Forum Geom.*, 1 (2001) 125–132.

[2] P. Yiu, The uses of homogeneous barycentric coordinates in plane euclidean geometry, *Int. J. Math. Educ. Sci. Technol.*, 31 (2000) 569–578.

[3] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University lecture notes, 2001.

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