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The M-Configuration of a Triangle

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Abstract. We give an easy construction of points A_a, B_a, C_a on the sides of a triangle ABC such that the figure M path $BC_aA_aB_aC$ consists of 4 segments of equal lengths. We study the configuration consisting of the three figures M of a triangle, and define an interesting mapping of triangle centers associated with such an M-configuration.

1. Introduction

Given a triangle ABC, we consider points A_a on the line BC, B_a on the half line CA, and C_a on the half line BA such that $BC_a = C_aA_a = A_aB_a = B_aC$. We shall refer to $BC_aA_aB_aC$ as M$_a$, because it looks like the letter M when triangle ABC is acute-angled. See Figures 1a. Figure 1b illustrates the case when the triangle is obtuse-angled. Similarly, we also have M$_b$ and M$_c$. The three figures M$_a$, M$_b$, M$_c$ constitute the M-configuration of triangle ABC. See Figure 2.

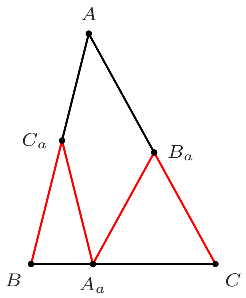

Figure 1a

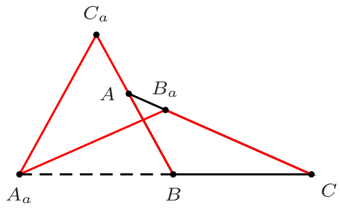

Figure 1b

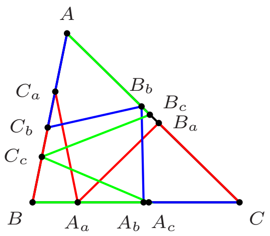

Figure 2

Proposition 1. *The lines AA_a, BB_a, CC_a concur at the point with homogeneous barycentric coordinates*

$$\left(\frac{1}{\cos A} : \frac{1}{\cos B} : \frac{1}{\cos C}\right).$$

Proof. Let l_a be the length of $BC_a = C_aA_a = A_aB_a = B_aC$. It is clear that the directed length $BA_a = 2l_a \cos B$ and $A_aC = 2l_a \cos C$, and $BA_a : A_aC = \cos B : \cos C$. For the same reason, $CB_b : B_bA = \cos C : \cos A$ and $AC_c : C_cB = \cos A : \cos B$. It follows by Ceva's theorem that the lines AA_a, BB_a, CC_a concur at the point with homogeneous barycentric coordinates given above.[1] □

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[1]This point appears in [3] as X_{92}.

Remark. Since $2l_a \cos B + 2l_a \cos C = a = 2R \sin A$, where R is the circumradius of triangle ABC,

$$l_a = \frac{a}{2(\cos B + \cos C)} = \frac{R \sin A}{\cos B + \cos C} = \frac{R \cos \frac{A}{2}}{\cos \frac{B-C}{2}}. \tag{1}$$

For later use, we record the absolute barycentric coordinates of A_a, B_a, C_a in terms of l_a:

$$\begin{aligned} A_a =& \frac{2l_a}{a}(\cos C \cdot B + \cos B \cdot C), \\ B_a =& \frac{1}{b}(l_a \cdot A + (b - l_a)C), \\ C_a =& \frac{1}{c}(l_a \cdot A + (c - l_a)B). \end{aligned} \tag{2}$$

2. Construction of M$_a$

Proposition 2. *Let A' be the intersection of the bisector of angle A with the circumcircle of triangle ABC.*

(a) *A_a is the intersection of BC with the parallel to AA' through the orthocenter H.*

(b) *B_a (respectively C_a) is the intersection of CA (respectively BA) with the parallel to CA' (respectively BA') through the circumcenter O.*

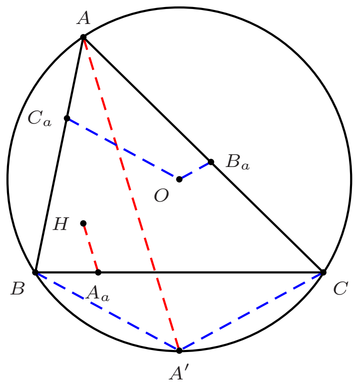

Figure 3a

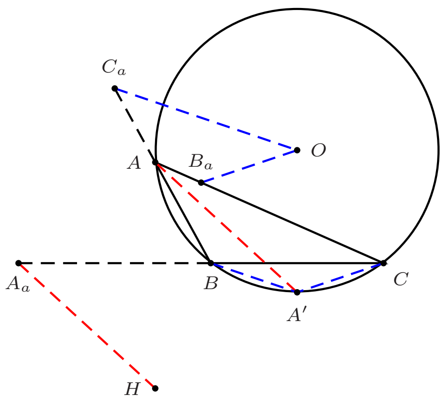

Figure 3b

Proof. (a) The line joining $A_a = (0 : \cos C : \cos B)$ to $H = \left(\frac{a}{\cos A} : \frac{b}{\cos B} : \frac{c}{\cos C}\right)$ has equation

$$\begin{vmatrix} 0 & \cos C & \cos B \\ \frac{a}{\cos A} & \frac{b}{\cos B} & \frac{c}{\cos C} \\ x & y & z \end{vmatrix} = 0.$$

This simplifies to

$$-(b-c)x\cos A + a(y\cos B - z\cos C) = 0.$$

It has infinite point

$$\begin{aligned}&(-a(\cos B+\cos C) : a\cos C-(b-c)\cos A : (b-c)\cos A + a\cos B)\\=&(-a(\cos B+\cos C) : b(1-\cos A) : c(1-\cos A)).\end{aligned}$$

It is clear that this is the same as the infinite point $(-(b+c) : b : c)$, which is on the line joining A to the incenter.

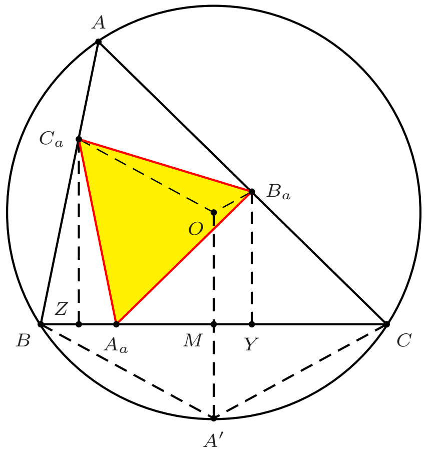

Figure 4

(b) Let M be the midpoint of BC, and Y, Z the pedals of B_a, C_a on BC. See Figure 4. We have

$$\begin{aligned}OM =&\frac{a}{2}\cot A = l_a(\cos B+\cos C)\cot A,\\C_aZ =&l_a\sin B,\\MZ =&\frac{a}{2} - l_a\cos B = l_a(\cos B+\cos C) - l_a\cos B = l_a\cos C.\end{aligned}$$

From this the acute angle between the line C_aO and BC has tangent ratio

$$\begin{aligned}\frac{C_aZ-OM}{MZ} =&\frac{\sin B-(\cos B+\cos C)\cot A}{\cos C}\\=&\frac{\sin B\sin A-(\cos B+\cos C)\cos A}{\cos C\sin A}\\=&\frac{-\cos(A+B)-\cos C\cos A}{\cos C\sin A} = \frac{\cos C(1-\cos A)}{\cos C\sin A}\\=&\frac{1-\cos A}{\sin A} = \tan\frac{A}{2}.\end{aligned}$$

It follows that C_aO makes an angle $\frac{A}{2}$ with the line BC, and is parallel to BA'. The same reasoning shows that B_aO is parallel to CA'. □

3. Circumcenters in the M-configuration

Note that $\angle B_aA_aC_a = \angle A$. It is clear that the circumcircles of $B_aA_aC_a$ and B_aAC_a are congruent. The circumradius is

$$R_a = \frac{l_a}{2\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)} = \frac{l_a}{2\cos\frac{A}{2}} = \frac{R}{2\cos\frac{B-C}{2}} \tag{3}$$

from (1).

Proposition 3. *The circumcircle of triangle AB_aC_a contains* (i) *the circumcenter O of triangle ABC,* (ii) *the orthocenter H_a of triangle $A_aB_aC_a$, and* (iii) *the midpoint of the arc BAC.*

Proof. (i) is an immediate corollary of Proposition 2(b) above.

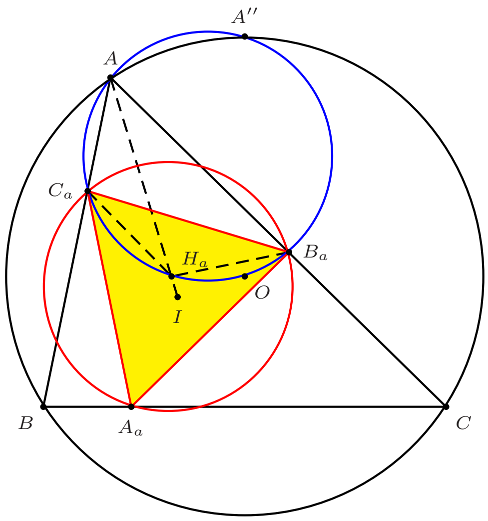

Figure 5

(ii) Let H_a be the orthocenter of triangle $A_aB_aC_a$. It is clear that

$$\angle B_aH_aC_a = \pi - \angle B_aA_aC_a = \pi - \angle BAC = \pi - \angle C_aAB_a.$$

It follows that H_a lies on the circumcircle of AB_aC_a. See Figure 5. Since the triangle $A_aB_aC_a$ is isosceles, $B_aH_a = C_aH_a$, and the point H_a lies on the bisector of angle A.

(iii) Let A'' be the midpoint of the arc BAC. By a simple calculation, $\angle AA''O = \frac{\pi}{2} - \frac{1}{2}|B - C|$. Also, $\angle AC_aO = \frac{\pi}{2} + \frac{1}{2}|B - C|$.[2] This shows that A'' also lies on the circle AB_aOC_a. □

The points B_a and C_a are therefore the intersections of the circle OAA'' with the sidelines AC and AB. This furnishes another simple construction of the figure M_a.

[2]This is $C + \frac{A}{2}$ if $C \geq B$ and $B + \frac{A}{2}$ otherwise.

Remarks. (1) If we take into consideration also the other figures M_b and M_c, we have three triangles AB_aC_a, BC_bA_b, CA_cB_c with their circumcircles intersecting at O.

(2) We also have three triangles A_aB_aCa, $A_bB_bC_b$, $A_cB_cC_c$ with their orthocenters forming a triangle perspective with ABC at the incenter I.

Proposition 4. *The circumcenter O_a of triangle $A_aB_aC_a$ is equidistant from O and H.*

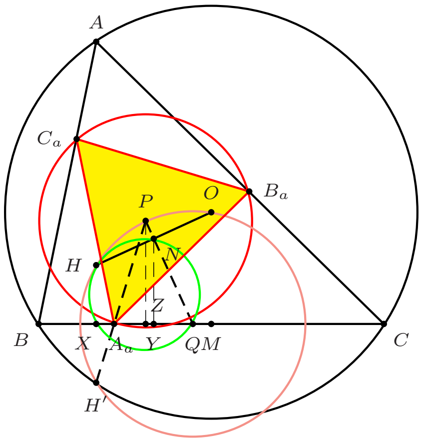

Figure 6

Proof. Construct the circle through O and H with center Q on the line BC. We prove that the midpoint P of the arc OH on the opposite side of Q is the circumcenter O_a of triangle $A_aB_aC_a$. See Figure 6. It will follow that O_a is equidistant from O and H. Let N be the midpoint of OH. Suppose the line PQ makes an angle φ with BC. Let X, Y, and M be the pedals of H, N, O on the line BC.

Since H, X, Q, N are concyclic, and the diameter of the circle containing them is $QH = \frac{NX}{\sin\varphi} = \frac{R}{2\sin\varphi}$. This is the radius of the circle OPH.

By symmetry, the circle OPH contains the reflection H' of H in the line BC.

$$\angle HH'P = \frac{1}{2}\angle HQP = \frac{1}{2}\angle HQN = \frac{1}{2}\angle HXN = \frac{1}{2}|B-C|.$$

Therefore, the angle between $H'P$ and BC is $\frac{\pi}{2} - \frac{1}{2}|B-C|$. It is obvious that the angle between A_aO_a and BC is the same. But from Proposition 2(a), the angle between HA_a and BC is the same too, so is the angle between the reflection $H'A_a$ and BC. From these we conclude that H', A_a, O_a and P are collinear. Now, let Z be the pedal of P on BC.

$$A_aP = \frac{PZ}{\cos\frac{1}{2}(B-C)} = \frac{QP\sin\varphi}{\cos\frac{1}{2}(B-C)} = \frac{R}{2\cos\frac{1}{2}(B-C)} = R_a.$$

Therefore, P is the circumcenter O_a of triangle $A_aB_aC_a$. $\square$

Applying this to the other two figures M_b and M_c, we obtain the following remarkable theorem about the M-configuration of triangle ABC.

Theorem 5. *The circumcenters of triangles $A_aB_aC_a$, $A_bB_bC_b$, and $A_cB_cC_c$ are collinear. The line containing them is the perpendicular bisector of the segment OH.*

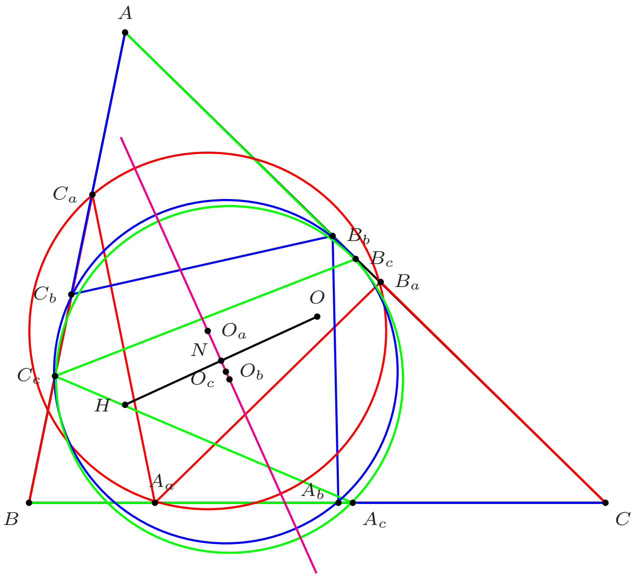

Figure 7

One can check without much effort that in homogeneous barycentric coordinates, the equation of this line is

$$\frac{\sin 3A}{\sin A}x + \frac{\sin 3B}{\sin B}y + \frac{\sin 3C}{\sin C}z = 0.$$

4. A central mapping

Let P be a triangle center in the sense of Kimberling [2, 3], given in homogeneous barycentric coordinates $(f(a,b,c) : f(b,c,a) : f(c,a,b))$ where $f = f_P$ satisfies $f(a,b,c) = f(a,c,b)$. If the reference triangle ABC is isosceles, say, with $AB = AC$, then P lies on the perpendicular bisector of BC and has coordinates of the form $(g_P : 1 : 1)$. The coordinate g depends only on the shape of the isosceles triangle, and we express it as a function of the *base* angle. We shall call $g = g_P$ the *isoscelized form* of the triangle center function f_P. Let P^* denote the isogonal conjugate of P.

Lemma 6. $g_{P^*}(B) = \frac{4\cos^2 B}{g_P(B)}$.

Proof. If $P = (g_P(B) : 1 : 1)$ for an isosceles triangle ABC with $B = C$, then

$$P^* = \left(\frac{\sin^2 A}{g_P(B)} : \sin^2 B : \sin^2 B\right) = \left(\frac{4\cos^2 B}{g_P(B)} : 1 : 1\right)$$

since $\sin^2 A = \sin^2(\pi - 2B) = \sin^2 2B = 4\sin^2 B\cos^2 B$. $\square$

Here are some examples.

Center	f_P	g_P
centroid	1	1
incenter	a	$2\cos B$
circumcenter	$a^2(b^2+c^2-a^2)$	$-2\cos 2B$
orthocenter	$\frac{1}{b^2+c^2-a^2}$	$\frac{-2\cos^2 B}{\cos 2B}$
symmedian point	a^2	$4\cos^2 B$
Gergonne point	$\frac{1}{s-a}$	$\frac{\cos B}{1-\cos B}$
Nagel point	$s-a$	$\frac{1-\cos B}{\cos B}$
Mittenpunkt	$a(s-a)$	$2(1-\cos B)$
Spieker point	$b+c$	$\frac{2}{1+2\cos B}$
X_{55}	$a^2(s-a)$	$4\cos B(1-\cos B)$
X_{56}	$\frac{a^2}{s-a}$	$\frac{4\cos^3 B}{1-\cos B}$
X_{57}	$\frac{a}{s-a}$	$\frac{2\cos^2 B}{1-\cos B}$

Consider a triangle center given by a triangle center function with isoscelized form $g = g_P$. The triangle center of the isosceles triangle C_aBA_a is the point $P_{a,b}$ with coordinates $(g(B) : 1 : 1)$ relative to C_aBA_a. Making use of the absolute barycentric coordinates of A_a, B_a, C_a given in (2), it is easy to see that this is the point

$$P_{a,b} = \left(\frac{g(B)l_a}{c} : \frac{g(B)(c-l_a)}{c} + 1 + \frac{2l_a}{a}\cos C : \frac{2l_a}{a}\cos B\right).$$

The same triangle center of the isosceles triangle B_aA_aC is the point

$$P_{a,c} = \left(\frac{g(C)l_a}{b} : \frac{2l_a}{a}\cos C : \frac{g(C)(b-l_a)}{b} + \frac{2l_a}{a}\cos B + 1\right).$$

It is clear that the lines $BP_{a,b}$ and $CP_{a,c}$ intersect at the point

$$\begin{aligned} P_a &= \left(\frac{g(B)g(C)l_a^2}{bc} : \frac{2g(B)l_a^2\cos C}{ca} : \frac{2g(C)l_a^2\cos B}{ab}\right) \\ &= (ag(B)g(C) : 2bg(B)\cos C : 2cg(C)\cos B) \\ &= \left(\frac{ag(B)g(C)}{2\cos B\cos C} : \frac{bg(B)}{\cos B} : \frac{cg(C)}{\cos C}\right). \end{aligned}$$

Figure 8 illustrates the case of the Gergonne point.

In the M-configuration, we may also consider the same triangle center (given in isoscelized form g_P of the triangle center function) in the isosceles triangles . These are the point $P_{b,c}$, $P_{b,a}$, $P_{c,a}$, $P_{c,b}$. The pairs of lines $CP_{b,c}$, $AP_{b,a}$ intersecting at P_b and $AP_{c,a}$, $BP_{c,b}$ intersecting at P_c. The coordinates of P_b and P_c can be

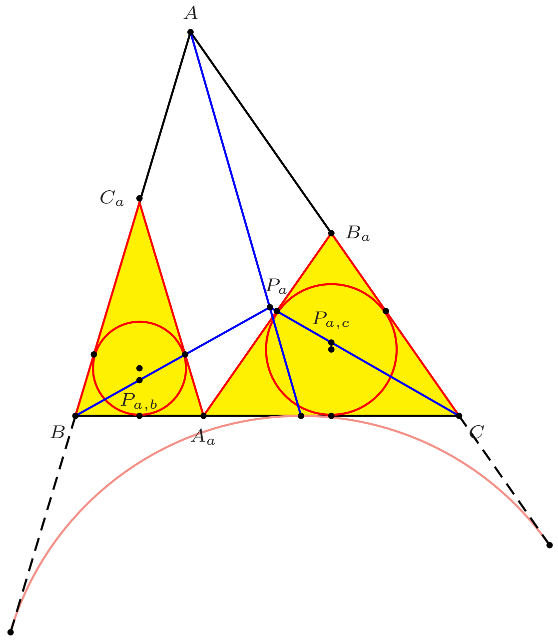

Figure 8

written down easily from those of P_a. From these coordinates, we easily conclude that that $P_aP_bP_c$ is perspective with triangle ABC at the point

$$\begin{aligned}\Phi(P) &= \left(\frac{ag_P(A)}{\cos A} : \frac{bg_P(B)}{\cos B} : \frac{cg_P(C)}{\cos C}\right)\\ &= (g_P(A)\tan A : g_P(B)\tan B : g_P(C)\tan C)\,.\end{aligned}$$

Proposition 7. $\Phi(P^*) = \Phi(P)^*$.

Proof. We make use of Lemma 6.

$$\begin{aligned}\Phi(P^*) =&(g_{P^*}(A)\tan A : g_{P^*}(B)\tan B : g_{P^*}(C)\tan C)\\ =&\left(\frac{4\cos^2 A}{g_P(A)}\tan A : \frac{4\cos^2 B}{g_P(B)}\tan B : \frac{4\cos^2 C}{g_P(C)}\tan C\right)\\ =&\left(\frac{\sin^2 A}{g_P(A)\tan A} : \frac{\sin^2 B}{g_P(B)\tan B} : \frac{\sin^2 C}{g_P(C)\tan C}\right)\\ =&\Phi(P)^*.\end{aligned}$$

□

We conclude with some examples.

P	$\Phi(P)$	P^*	$\Phi(P^*) = \Phi(P)^*$
incenter	incenter		
centroid	orthocenter	symmedian point	circumcenter
circumcenter	X_{24}	orthocenter	X_{68}
Gergonne point	Nagel point	X_{55}	X_{56}
Nagel point	X_{1118}	X_{56}	$X_{1259} = X^*_{1118}$
Mittenpunkt	X_{34}	X_{57}	$X_{78} = X^*_{34}$

For the Spieker point, we have

$$\begin{aligned}\Phi(X_{10}) &= \left(\frac{\tan A}{1+2\cos A} : \frac{\tan B}{1+2\cos B} : \frac{\tan C}{1+2\cos C}\right)\\ &= \left(\frac{1}{a(b^2+c^2-a^2)(b^2+c^2-a^2+bc)} : \cdots : \cdots\right).\end{aligned}$$

This triangle center does not appear in the current edition of [3].

Remark. For $P = X_8$, the Nagel point, the point P_a has an alternative description. Antreas P. Hatzipolakis [1] considered the incircle of triangle ABC touching the sides CA and AB at Y and Z respectively, and constructed perpendiculars from Y, Z to BC intersecting the incircle again at Y' and Z'. See Figure 9. It happens that B, Z', $P_{a,b}$ are collinear; so are C, Y', $P_{a,c}$. Therefore, BZ' and CY' intersect at P_a. The coordinates of Y' and Z' are

$$\begin{aligned}Y' =&(a^2(b+c-a)(c+a-b) : (a^2+b^2-c^2)^2 : (b+c)^2(a+b-c)(c+a-b)),\\ Z' =&(a^2(b+c-a)(a+b-c) : (b+c)^2(c+a-b)(a+b-c) : (a^2-b^2+c^2)^2).\end{aligned}$$

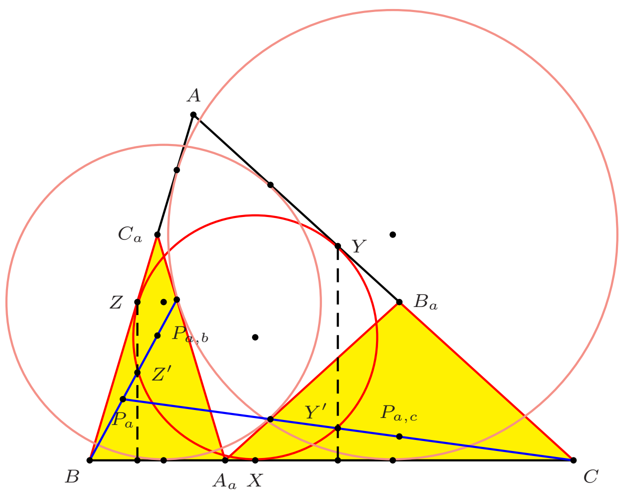

Figure 9

The lines BZ' and CY' intersect at

$$P_a = \left(a^2(b+c-a) : \frac{(a^2+b^2-c^2)^2}{c+a-b} : \frac{(a^2-b^2+c^2)^2}{a+b-c} \right)$$

$$= \left(\frac{a^2(b+c-a)}{(a^2-b^2+c^2)^2(a^2+b^2-c^2)^2} : \frac{1}{(c+a-b)(a^2-b^2+c^2)^2} : \frac{1}{(a+b-c)(a^2+b^2-c^2)^2} \right).$$

It was in this context that Hatzipolakis constructed the triangle center

$$X_{1118} = \left(\frac{1}{(b+c-a)(b^2+c^2-a^2)^2} : \cdots : \cdots \right).$$

References

[1] A. P. Hatzipolakis, Hyacinthos message 5321, April 30, 2002.

[2] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.

[3] C. Kimberling, *Encyclopedia of Triangle Centers*, May 23, 2003 edition available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

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