Forum Geometricorum
Volume 3 (2003) 29–34.

FORUM GEOM
ISSN 1534-1178

On the Procircumcenter and Related Points

Alexei Myakishev

Abstract. Given a triangle ABC, we solve the construction problem of a point P, together with points B_c, C_b on BC, C_a, A_c on CA, and A_b, B_a on AB such that PB_aC_a, A_bPC_b, and A_cB_cP are congruent triangles similar to ABC. There are altogether seven such triads. If these three congruent triangles are all oppositely similar to ABC, then P must be the procircumcenter, with trilinear coordinates $(a^2 \cos A : b^2 \cos B : c^2 \cos C)$. If at least one of the triangles in the triad is directly similar to ABC, then P is either a vertex or the midpoint of a side of the tangential triangle. We also determine the ratio of similarity in each case.

1. Introduction

Given a triangle ABC, we consider the construction of a point P, together with points B_c, C_b on BC, C_a, A_c on CA, and A_b, B_a on AB such that PB_aC_a, A_bPC_b, and A_cB_cP are congruent triangles similar to ABC. We first consider in §§2,3 the case when these triangles are all *oppositely* similar to ABC. See Figure 1. In §4, the possibilities when at least one of these congruent triangles is directly similar to ABC are considered. See, for example, Figure 2.

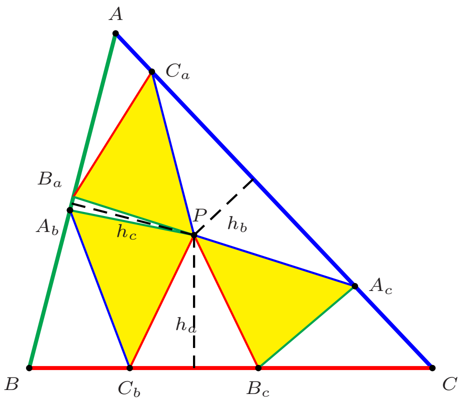

Figure 1

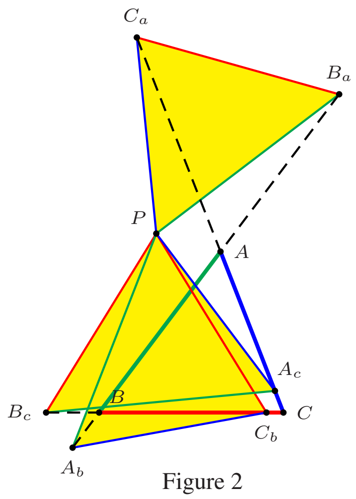

Figure 2

Publication Date: January 27, 2003. Communicating Editor: Jean-Pierre Ehrmann.

The author thanks Jean-Pierre Ehrmann and Paul Yiu for their helps in the preparation of this paper.

2. The case of opposite similarity: construction of P

With reference to Figure 1, we try to find the trilinear coordinates of P. As usual, we denote the lengths of the sides opposite to angles A, B, C by a, b, c. Denote the *oriented* angles C_bPB_c by φ_a, A_cPC_a by φ_b, and B_aPA_b by φ_c. [1] Since $PC_b = PB_c$, $\angle PB_cC_b = \frac{1}{2}(\pi - \varphi_a)$. Since also $\angle PB_cA_c = B$, we have $\angle A_cB_cC = \frac{1}{2}(\pi + \varphi_a) - B$. For the same reason, $\angle B_cA_cC = \frac{1}{2}(\pi + \varphi_b) - A$. Considering the sum of the angles in triangle A_cB_cC, we have $\frac{1}{2}(\varphi_a + \varphi_b) = \pi - 2C$. Since $\varphi_a + \varphi_b + \varphi_c = \pi$, we have $\varphi_c = 4C - \pi$. Similarly, $\varphi_a = 4A - \pi$ and $\varphi_b = 4B - \pi$.

Let k be the ratio of similarity of the triangles PB_aC_a, A_bPC_b, and A_cB_cP with ABC, *i.e.*, $B_aC_a = PC_b = B_cP = k \cdot BC = ka$. The perpendicular distance from P to the line BC is

$$h_a = ka\cos\frac{\varphi_a}{2} = ka\cos\left(2A - \frac{\pi}{2}\right) = ka\sin 2A.$$

Similarly, the perpendicular distances from P to CA and AB are $h_b = kb\sin 2B$ and $h_c = kc\sin 2C$. It follows that P has trilinear coordinates,

$$(a\sin 2A : b\sin 2B : c\sin 2C) \sim (a^2\cos A : b^2\cos B : c^2\cos C). \qquad (1)$$

Note that we have found not only the trilinears of P, but also the angles of isosceles triangles PC_bB_c, PA_cC_a, PB_aA_b. It is therefore easy to construct the triangles by ruler and compass from P. Now, we easily identify P as the isogonal conjugate of the isotomic conjugate of the circumcenter O, which has trilinear coordinates $(\cos A : \cos B : \cos C)$. We denote this point by $\overline{O}$ and follow John H. Conway in calling it the *procircumcenter* of triangle ABC. We summarize the results in the following proposition.

Proposition 1. *Given a triangle ABC* not *satisfying (2), the point P for which there are congruent triangles PB_aC_a, A_bPC_b, and A_cB_cP oppositely similar to ABC (with B_c, C_b on BC, C_a, A_c on CA, and A_b, B_a on AB) is the procircumcenter $\overline{O}$. This is a finite point unless the given triangle satisfies*

$$a^4(b^2 + c^2 - a^2) + b^4(c^2 + a^2 - b^2) + c^4(a^2 + b^2 - c^2) = 0. \qquad (2)$$

The procircumcenter $\overline{O}$ appears as X_{184} in [3], and is identified as the inverse of the Jerabek center X_{125} in the Brocard circle. A simple construction of $\overline{O}$ is made possible by the following property discovered by Fred Lang.

Proposition 2 (Lang [4])**.** *Let the perpendicular bisectors of BC, CA, AB intersect the other pairs of sides at B_1, C_1, C_2, A_2, A_3, B_3 respectively. The perpendicular bisectors of B_1C_1, C_2A_2 and A_3B_3 bound a triangle homothetic to ABC at the procircumcenter $\overline{O}$.*

---

[1] We regard the orientation of triangle ABC as positive. The oriented angles are defined modulo 2π.

3. The case of opposite similarity: ratio of similarity

We proceed to determine the ratio of similarity k. We shall make use of the following lemmas.

Lemma 3. *Let $\triangle$ denote the area of triangle ABC, and R its circumradius.*
(1) $\triangle = 2R^2 \sin A \sin B \sin C$*;*
(2) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$*;*
(3) $\sin 4A + \sin 4B + \sin 4C = -4 \sin 2A \sin 2B \sin 2C$*;*
(4) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$.

Proof. (1) By the law of sines,

$$\triangle = \frac{1}{2} bc \sin A = \frac{1}{2}(2R \sin B)(2R \sin C) \sin A = 2R^2 \sin A \sin B \sin C.$$

For (2),

$$\begin{aligned} & \sin 2A + \sin 2B + \sin 2C \\ =& 2 \sin A \cos A + 2 \sin(B + C) \cos(B - C) \\ =& 2 \sin A(-\cos(B + C) + \cos(B - C)) \\ =& 4 \sin A \sin B \sin C. \end{aligned}$$

The proof of (3) is similar. For (4),

$$\begin{aligned} & \sin^2 A + \sin^2 B + \sin^2 C \\ =& \sin^2 A + 1 - \frac{1}{2}(\cos 2B + \cos 2C) \\ =& \sin^2 A + 1 - \cos(B + C) \cos(B - C) \\ =& 2 - \cos^2 A + \cos A \cos(B - C) \\ =& 2 + \cos A(\cos(B + C) + \cos(B - C)) \\ =& 2 + 2 \cos A \cos B \cos C. \end{aligned}$$

□

Lemma 4. $a^2 + b^2 + c^2 = 9R^2 - OH^2$*, where R is the circumradius, and O, H are respectively the circumcenter and orthocenter of triangle ABC.*

This was originally due to Euler. An equivalent statement

$$a^2 + b^2 + c^2 = 9(R^2 - OG^2),$$

where G is the centroid of triangle ABC, can be found in [2, p.175].

Proposition 5 (Dergiades [1])**.** *The ratio of similarity of $\overline{O}B_aC_a$, $A_b\overline{O}C_b$, and $A_cB_c\overline{O}$ with ABC is*

$$k = \left| \frac{R^2}{3R^2 - OH^2} \right|.$$

Proof. Since $2\triangle = a \cdot h_a + b \cdot h_b + c \cdot h_c$, and $h_a = ka \sin 2A$, $h_b = kb \sin 2B$, and $h_c = kc \sin 2C$, the ratio of similarity is the absolute value of

$$\begin{aligned}
& \frac{2\triangle}{a^2 \sin 2A + b^2 \sin 2B + c^2 \sin 2C} \\
= & \frac{4R^2 \sin A \sin B \sin C}{4R^2(\sin^2 A \sin 2A + \sin^2 B \sin 2B + \sin^2 C \sin 2C)} && \text{[Lemma 3(1)]} \\
= & \frac{2 \sin A \sin B \sin C}{(1-\cos 2A)\sin 2A + (1-\cos 2B)\sin 2B + (1-\cos 2C)\sin 2C} \\
= & \frac{4 \sin A \sin B \sin C}{2(\sin 2A + \sin 2B + \sin 2C) - (\sin 4A + \sin 4B + \sin 4C)} \\
= & \frac{4 \sin A \sin B \sin C}{8 \sin A \sin B \sin C + 4 \sin 2A \sin 2B \sin 2C} && \text{[Lemma 3(2,3)]} \\
= & \frac{1}{2 + 8 \cos A \cos B \cos C} \\
= & \frac{1}{4(\sin^2 A + \sin^2 B + \sin^2 C) - 6} && \text{[Lemma 3(4)]} \\
= & \frac{R^2}{a^2 + b^2 + c^2 - 6R^2} \\
= & \frac{R^2}{3R^2 - OH^2}
\end{aligned}$$

by Lemma 4. □

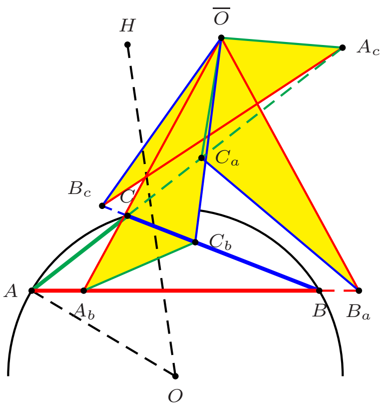

Figure 3: $OH = 2R$

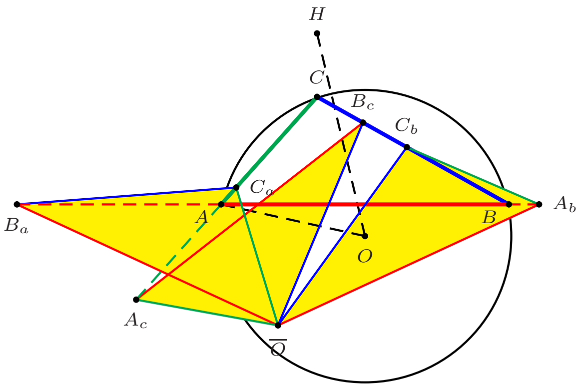

Figure 4: $OH = \sqrt{2}R$

From Proposition 5, we also infer that $\overline{O}$ is an infinite point if and only if $OH = \sqrt{3}R$. More interesting is that for triangles satisfying $OH = 2R$ or $\sqrt{2}R$, the congruent triangles in the triad are also congruent to the reference triangle ABC. See Figures 3 and 4. These are triangles satisfying

$$a^4(b^2 + c^2 - a^2) + b^4(c^2 + a^2 - b^2) + c^4(a^2 + b^2 - c^2) = \pm a^2 b^2 c^2.$$

4. Cases allowing direct similarity with ABC

As Jean-Pierre Ehrmann has pointed out, by considering all possible orientations of the triangles PB_aC_a, A_bPC_b, A_cB_cP, there are other points, apart from the procircumcenter $\overline{O}$, that yield triads of congruent triangles similar to ABC.

4.1. *Exactly one of the triangles oppositely similar to ABC.* Suppose, for example, that among the three congruent triangles, only PB_aC_a be oppositely similar to ABC, the other two, A_bPC_b and A_cB_cP being directly similar. We denote by P_a^+ the point P satisfying these conditions. Modifying the calculations in §2, we have

$$\varphi_a = \pi + 2A, \quad \varphi_b = \pi - 2A, \quad \varphi_c = \pi - 2A.$$

From these, we obtain the trilinears of P_a^+ as

$$(-a \sin A : b \sin A : c \sin A) = (-a : b : c).$$

It follows that P_a^+ is the A-vertex of the tangential triangle of ABC. See Figure 5.

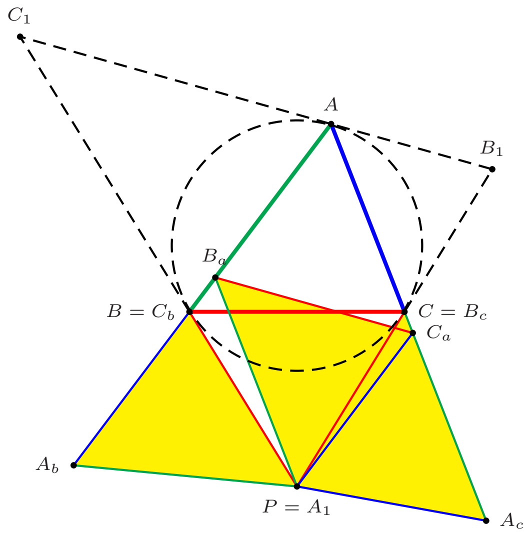

Figure 5

The ratio of similarity, by a calculation similar to that performed in §3, is $k = \left|\frac{1}{2\cos A}\right|$. This is equal to 1 only when $A = \frac{\pi}{3}$ or $\frac{2\pi}{3}$. In these cases, the three triangles are congruent to ABC.

Clearly, there are two other triads of congruent triangles corresponding to the other two vertices of the tangential triangle.

4.2. *Exactly one of triangles directly similar to ABC.* Suppose, for example, that among the three congruent triangles, only PB_aC_a be directly similar to ABC, the other two, A_bPC_b and A_cB_cP being oppositely similar. We denote by P_a^- the point P satisfying these conditions. See Figure 6. In this case, we have

$$\varphi_a = 2A - \pi, \quad \varphi_b = \pi + 2B - 2C, \quad \varphi_c = \pi + 2C - 2B.$$

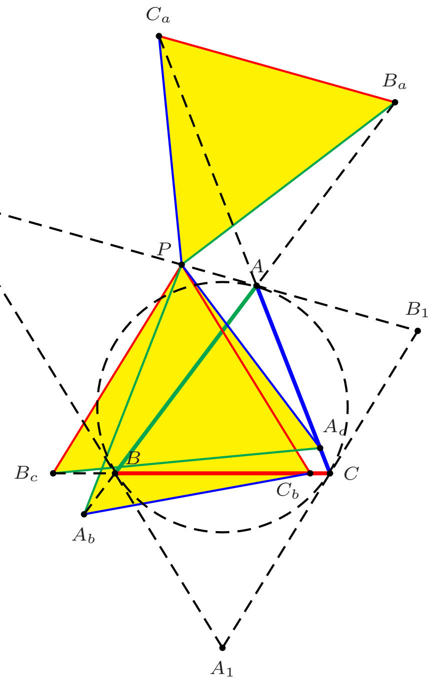

Figure 6

From these, we obtain the trilinears of P_a^- as

$$(-a\sin A : b\sin(B-C) : c\sin(C-B)) = (-a^3 : b(b^2-c^2) : c(c^2-b^2)).$$

It is easy to check that this is the midpoint of the side B_1C_1 of the tangential triangle of ABC. In this case, the ratio of similarity is $k = \left|\frac{1}{4\cos B\cos C}\right|$.

Clearly, there are two other triads of congruent triangles corresponding to the midpoints of the remaining two sides of the tangential triangle.

We conclude with the remark that it is not possible for all three of the congruent triangles to be directly similar to ABC, since this would require $\varphi_a = \varphi_b = \varphi_c = \pi$.

References

[1] N. Dergiades, Hyacinthos message 5437, May 10, 2002.
[2] R. A. Johnson, *Advanced Euclidean Geometry*, 1925, Dover reprint.
[3] C. Kimberling, Triangle centers and central triangles, *Congressus Numerantium*, 129 (1998) 1 – 285.
[4] F. Lang, Hyacinthos message 1190, August 13, 2000.

Alexei Myakishev: Smolnaia 61-2, 138, Moscow, Russia, 125445
E-mail address: alex.geom@mtu-net.ru